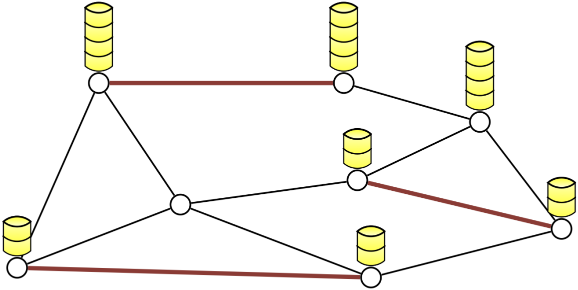

ADGA 2013

2nd Workshop on Advances on Distributed Graph Algorithms

Distributed Load Balancing on Graphs

Thomas Sauerwald

UNIVERSITY OF CAMBRIDGE

Outline

Introduction

Load Balancing on Hypercubes

Load Balancing on Arbitrary Graphs

Conclusions

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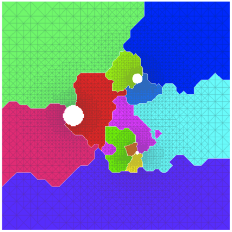
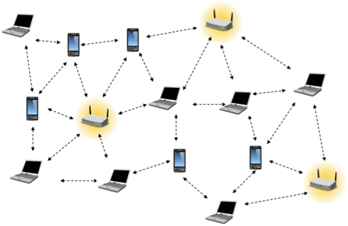
Load Balancing on Hypercubes

Load Balancing on Arbitrary Graphs

Conclusions

Load Balancing

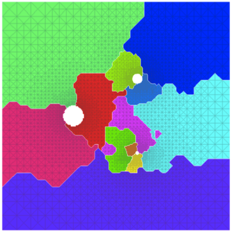


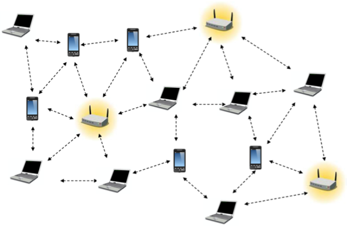


Applications

- Numerical Simulations
- Traffic in Communication Network
- Data Management in P2P network

Load Balancing





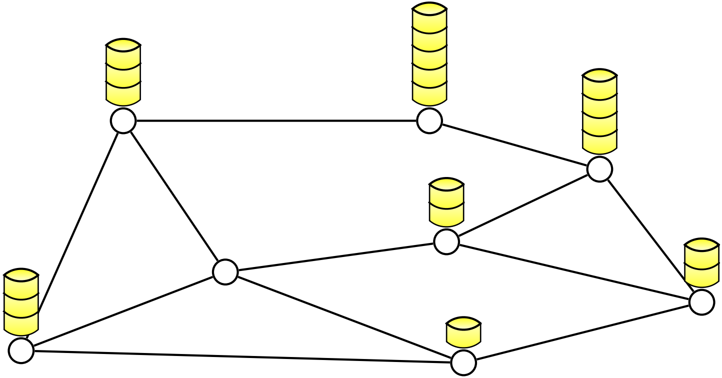
Applications

- Numerical Simulations
- Traffic in Communication Network
- Data Management in P2P network

Conditions

- network structure and load distribution unknown
- node can only communicate with neighbors

Discrete Load Balancing with Unit-Size-Token



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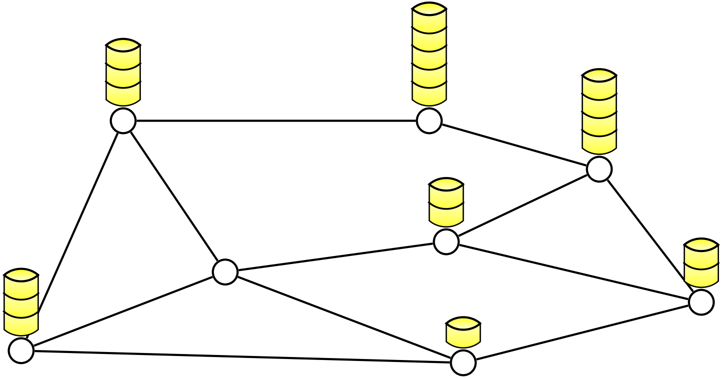

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For every round $t = 1, 2, 3, \ldots$

- Generate a matching
- Matched vertices average load

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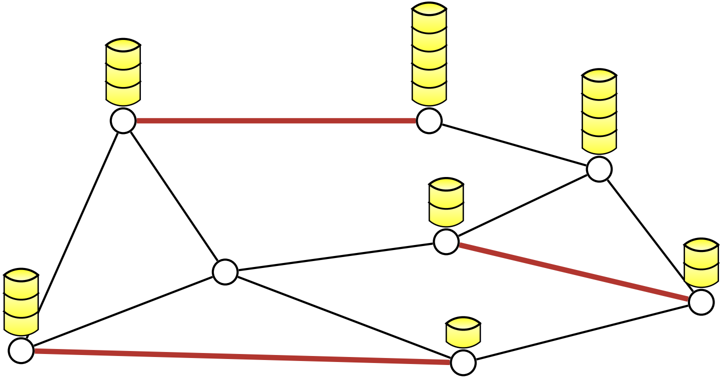

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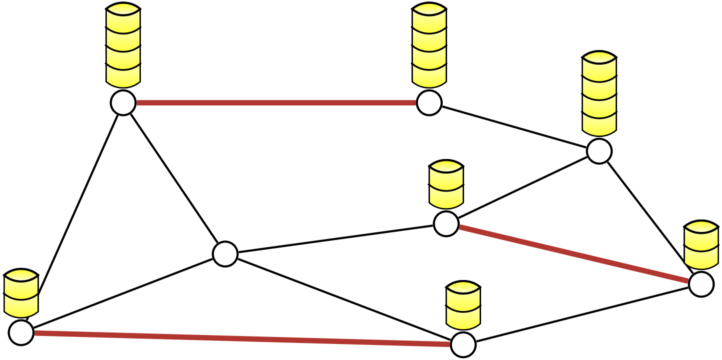

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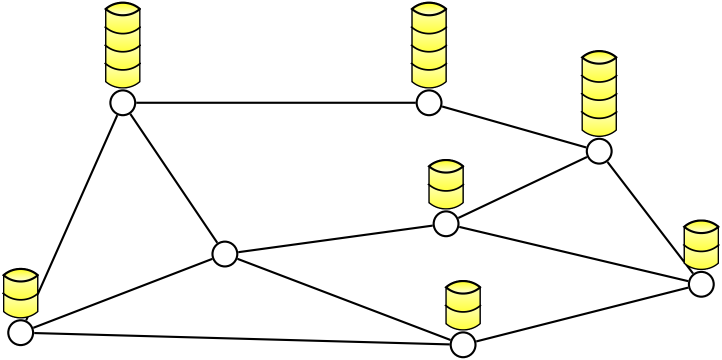

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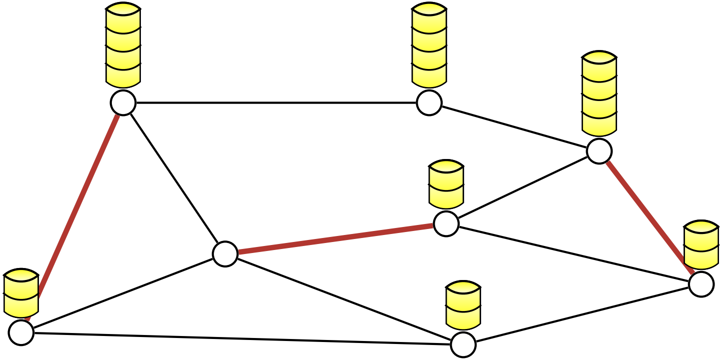

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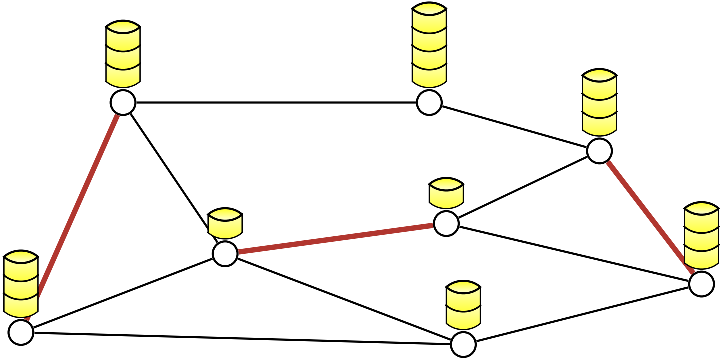

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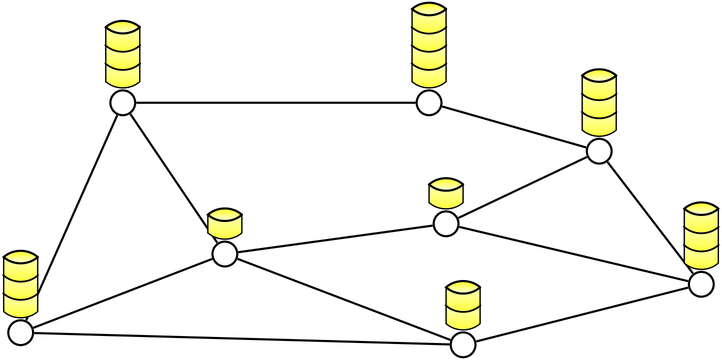

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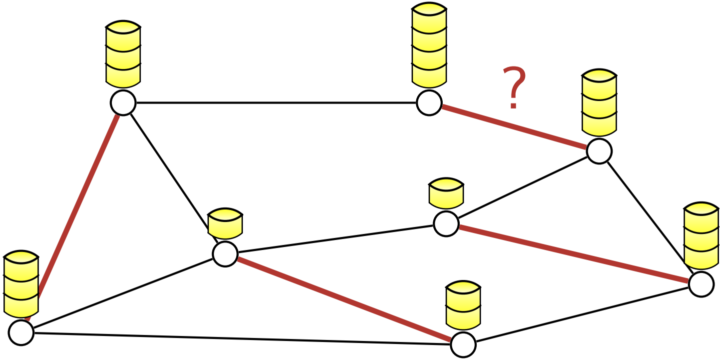

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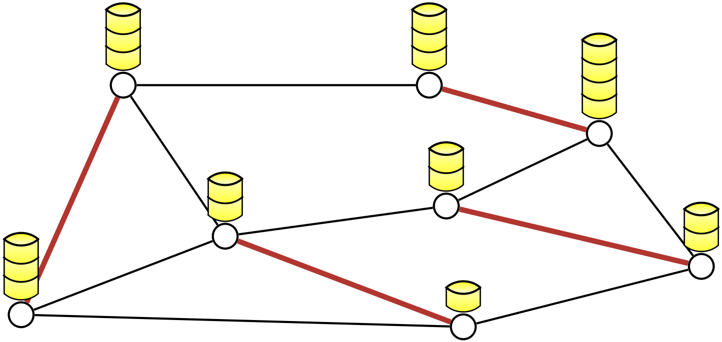

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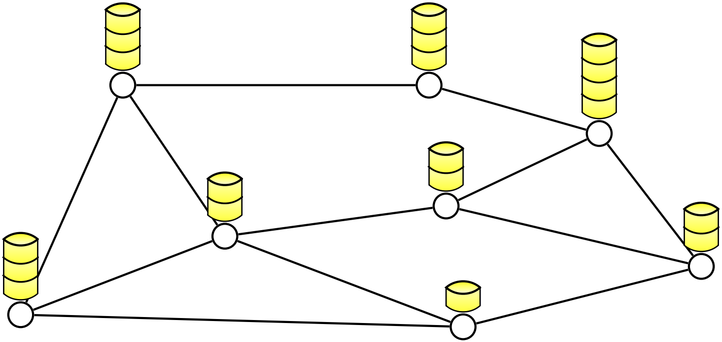
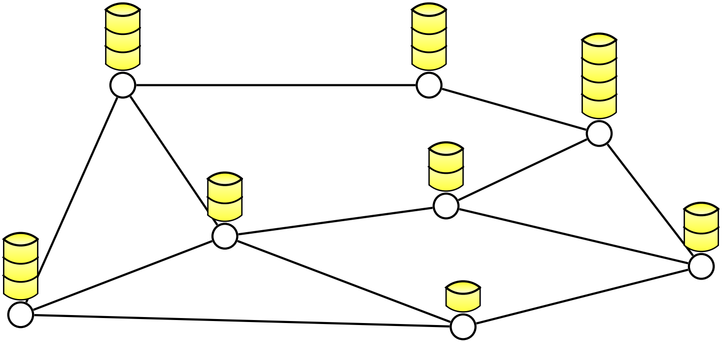


Conditions

- ✓ Graphs's structure and load is unknown to every node
- ✓ Nodes can only communicate with neighbors

Discrete Load Balancing with Unit-Size-Token

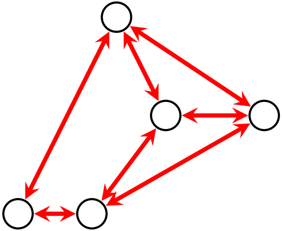


How should we generate the matchings?

Communication Models

Diffusion

+ natural
– high communication



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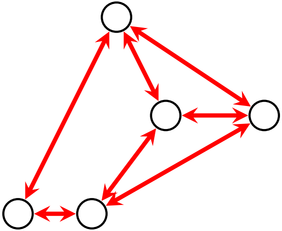
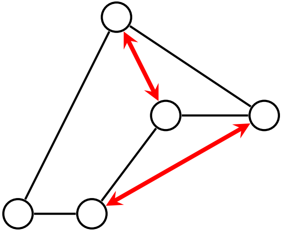
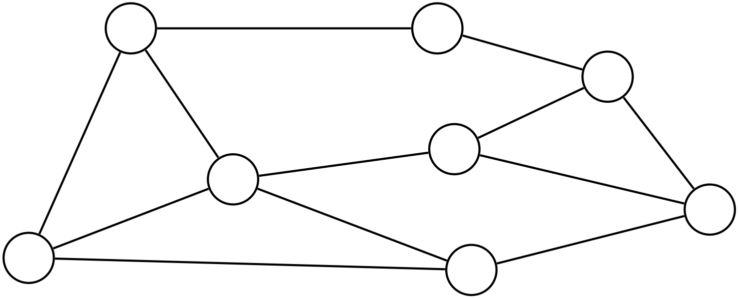


Matching Model
- + less communication
- + monotone
- – matchings have to be specified

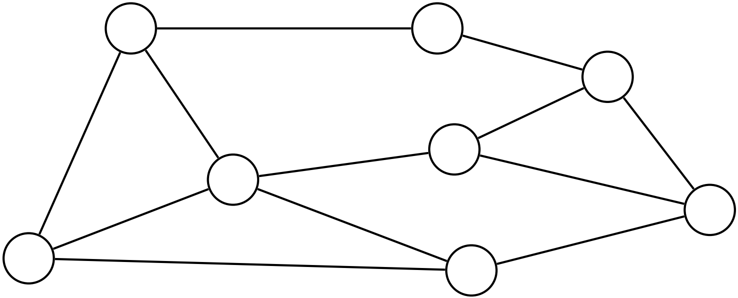


Generating Matching Using Edge Coloring



Balancing Circuit Model

Generating Matching Using Edge Coloring



Balancing Circuit Model

1. Take an edge coloring with $c \leqslant \text{maxdeg} + 1$ colors.

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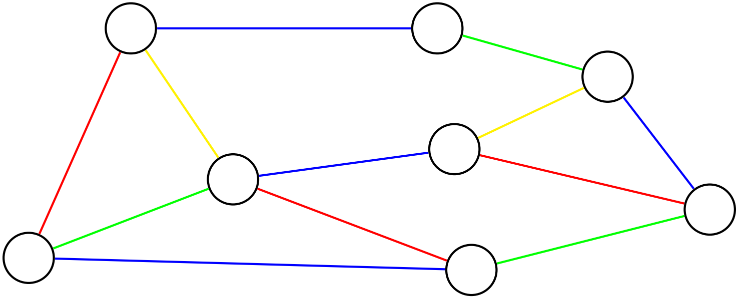

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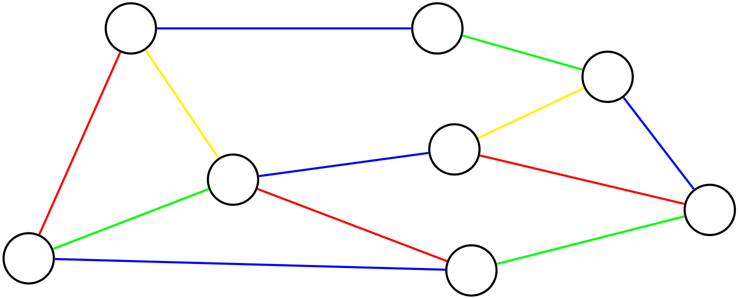

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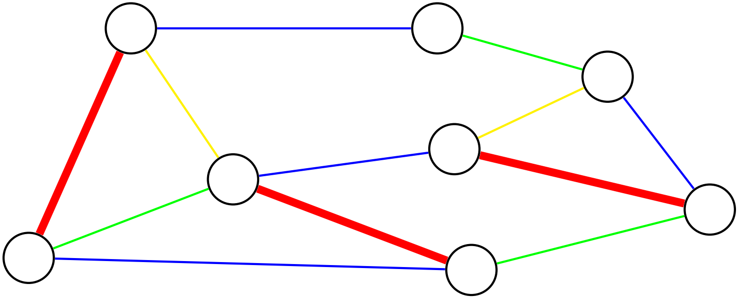

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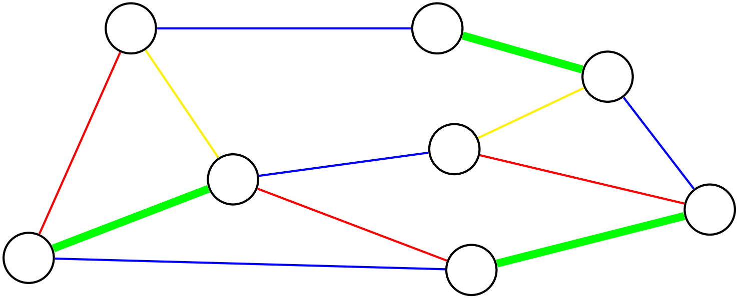

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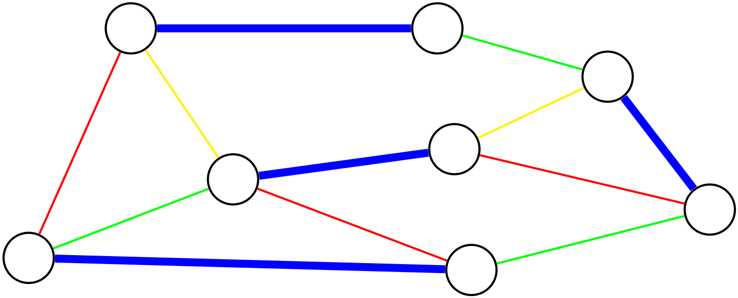

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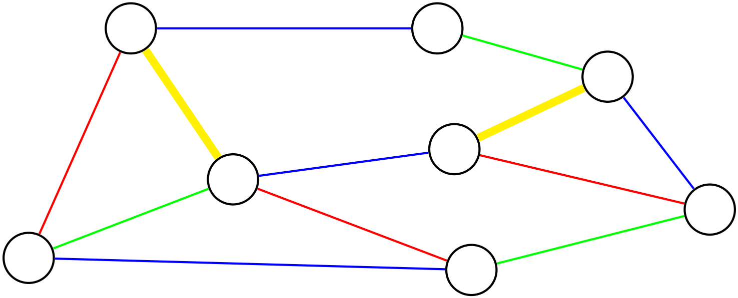

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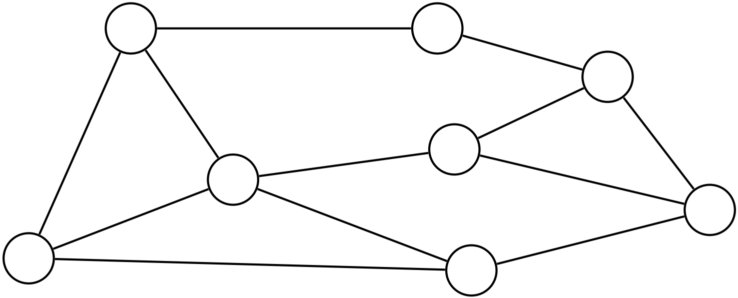
Generating Matching Using Randomization



Random Matching (Boyd et al., 2006)

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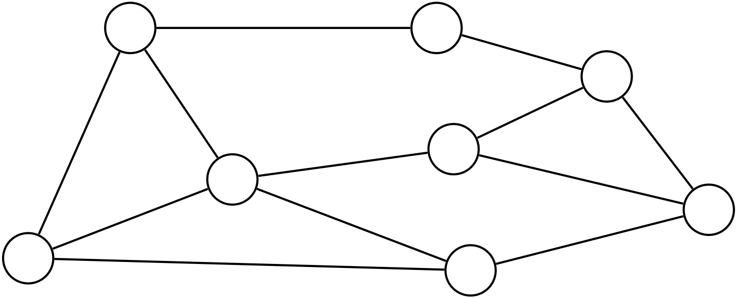

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1. First, every node becomes active (or passive) with prob. $1/2$.

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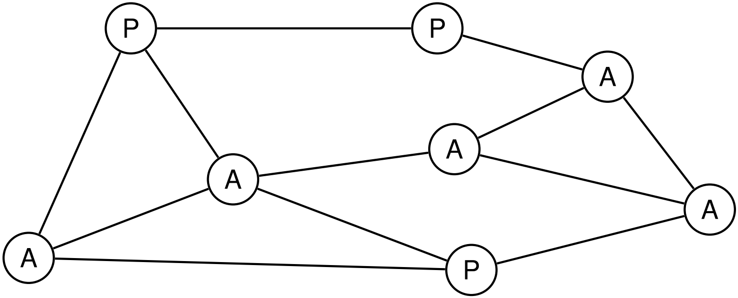

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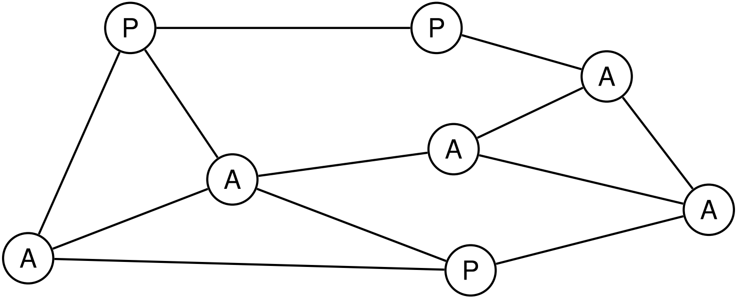

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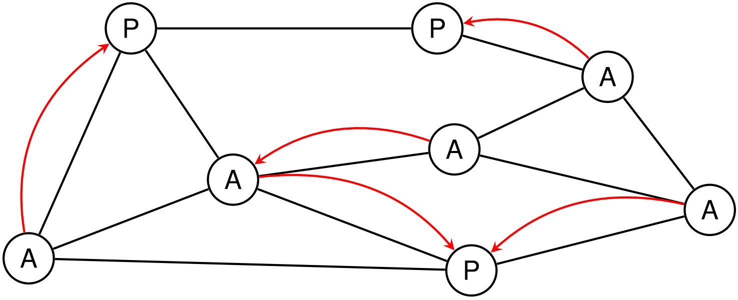

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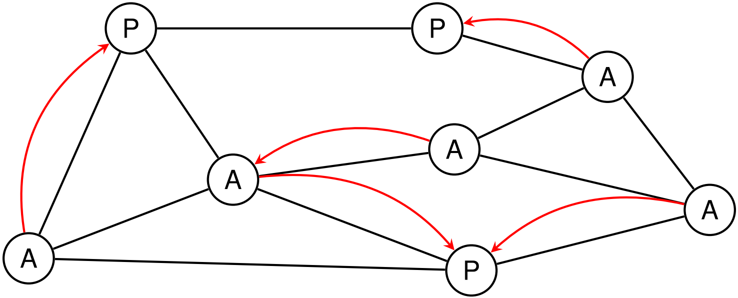

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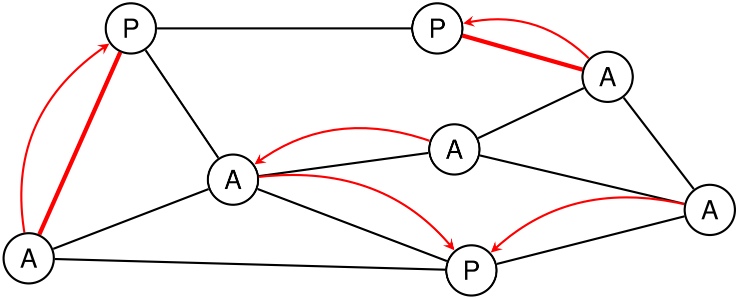

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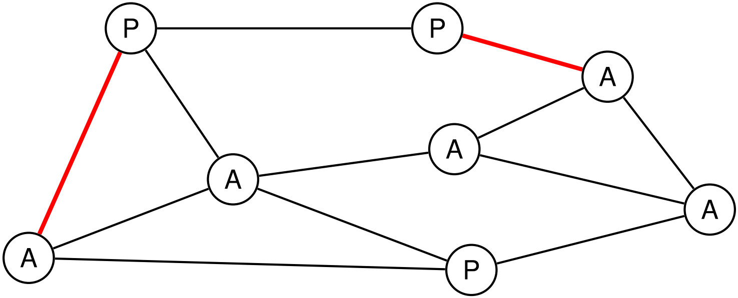

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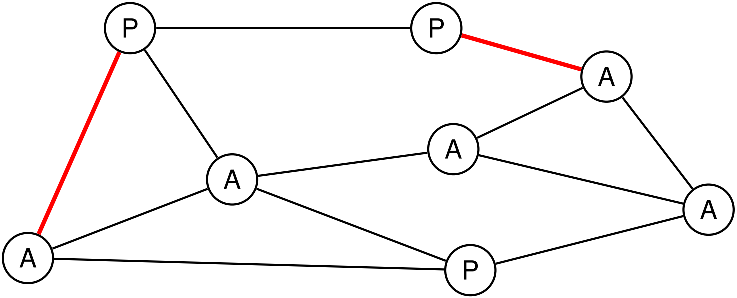


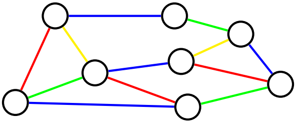
Crucial Properties:

- An edge $\{u, v\} \in E$ is included with prob. $\approx \frac{1}{\text{maxdeg}}$
- Matchings in different rounds are generated independently

Balancing Circuit vs. Random Matching

Balancing Circuit (Dimension Exchange)

- graphs with structure (grids and hypercubes)
- edge-coloring and order may affect convergence (dense graphs)



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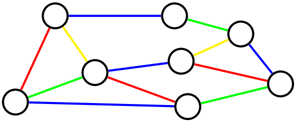
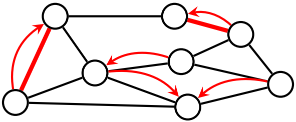


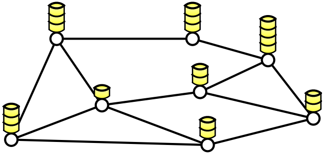
Random Matching

- applicable to any graph
- convergence captured by the spectral gap of the graph



Smoothness of the Load Distribution

- let $x \in \mathbb{R}^n$ be a load vector
- $\overline{x}$ denotes the average load



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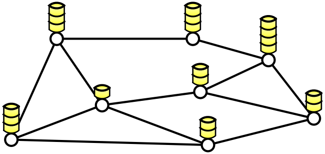
- ℓ_2-norm: $\|x - \overline{x}\|_2 = \sqrt{\sum_{i=1}^{n}(x_i - \overline{x})^2}$
- makespan: $\max_{i=1}^{n} x_i$
- discrepancy: $\max_{i=1}^{n} x_i - \min_{i=1}^{n} x_i$.



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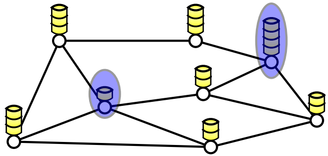

Random Matching in Continuous Case

Ghosh, Muthukrishnan, 1994

- Let $\Phi^t = \sum_{i=1}^{n} \left(x_i^t - \overline{x}\right)^2$. Then,

$$\mathbf{E}\left[\Phi^t - \Phi^{t+1}\right] \geqslant$$

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where $\lambda \in (0, 1]$ is the spectral expansion.

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$\Rightarrow$ For any initial load vector with discrepancy K, the discrepancy is at most ϵ w.p. $1 - n^{-1}$ after $\mathcal{O}\left(\frac{\log(n \cdot \frac{K}{\epsilon})}{1-\lambda}\right)$ rounds.

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- Even though load is moved only along a subset of edges, the convergence is in terms of the global properties

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Continuous case:

- Well understood and rapid convergence
- less realistic as tokens can be divided arbitrarily often

- Speed of convergence essentially the same as for FOS diffusion
- Even though load is moved only along a subset of edges, the convergence is in terms of the global properties

Discrete vs. Continuous Load Balancing

What is the relation between the discrete and continuous case?

Subramannian and Scherson 1994, Ghosh, Leighton, Maggs, Muthukrishnan, Plaxton, Rajaraman, Richa, Tarjan and Zuckerman 1995, Lovasz and Winkler 1995, Muthukrishnan, Ghosh and Schultz 1996, Rabani, Sinclair and Wanka 1998.

Outline

Introduction

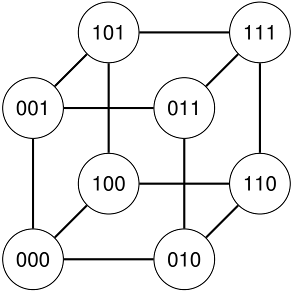
Load Balancing on Hypercubes

Load Balancing on Arbitrary Graphs

Conclusions

Hypercube

- d-dimensional hypercube
  - $V = \{0,1\}^d$, $n = 2^d$
  - $E = \{\{u, v\} \colon u \text{ and } v \text{ differ in one bit}\}$



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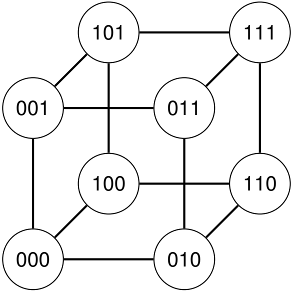
Dimension Exchange (Balancing Circuit)

- round i: every node communicates along dimension i
- load of communicating nodes is averaged



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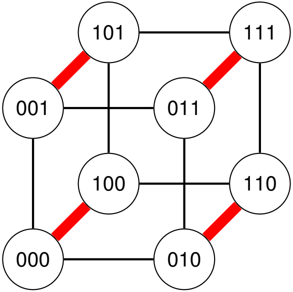
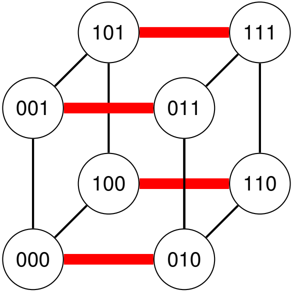


Hypercube

- d-dimensional hypercube
  - $V = \{0,1\}^d$, $n = 2^d$
  - $E = \{\{u,v\}: u \text{ and } v \text{ differ in one bit}\}$

Dimension Exchange (Balancing Circuit)

- round i: every node communicates along dimension i
- load of communicating nodes is averaged



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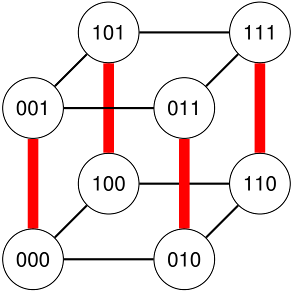
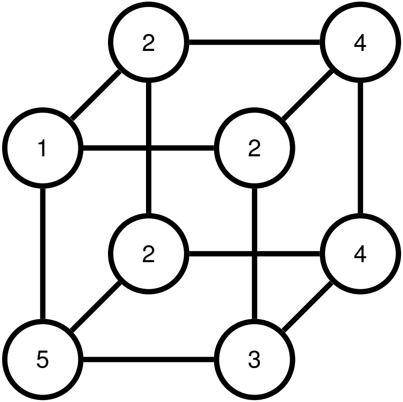


Continuous vs. Discrete Load Balancing



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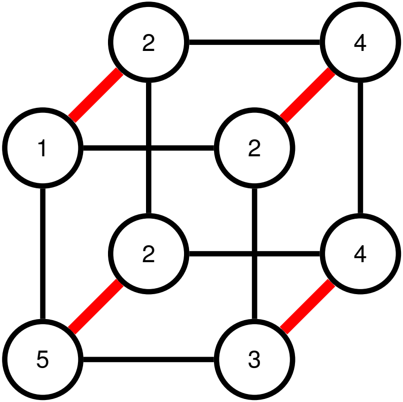

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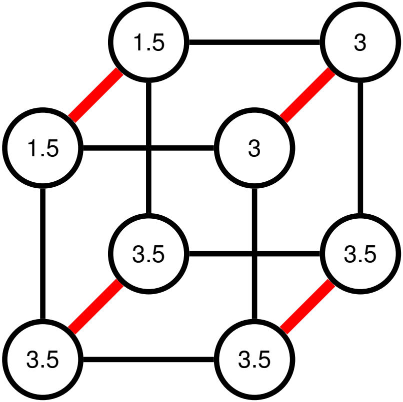

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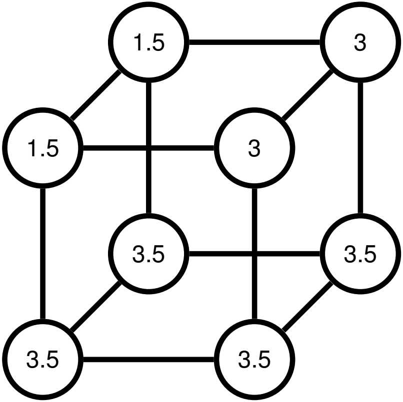

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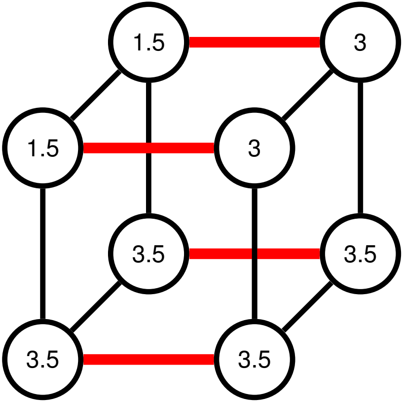

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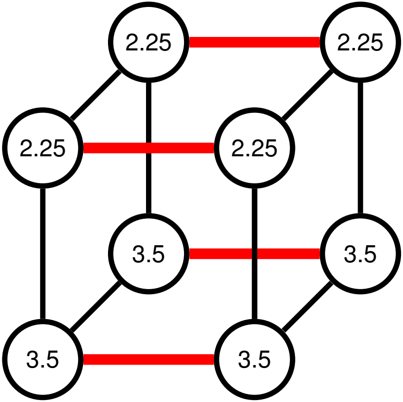

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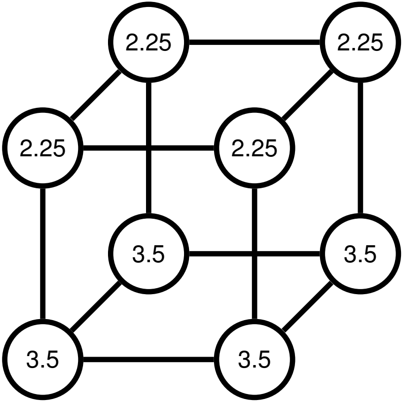

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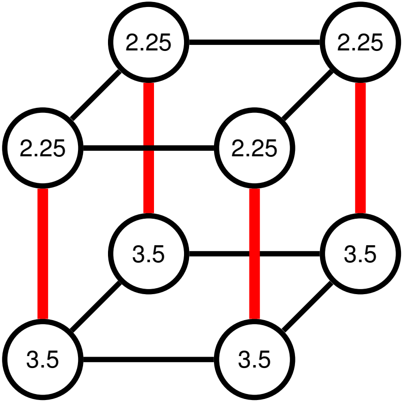

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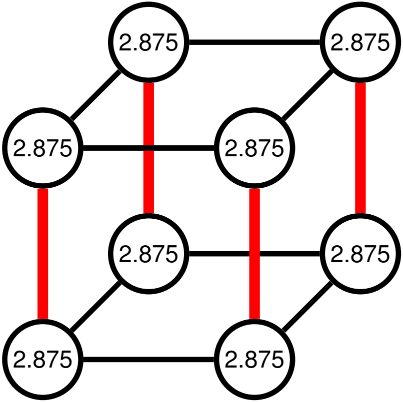

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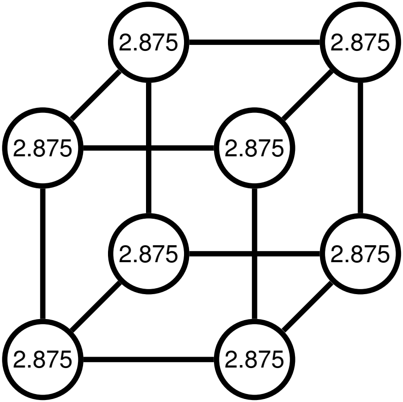

perfectly balanced!

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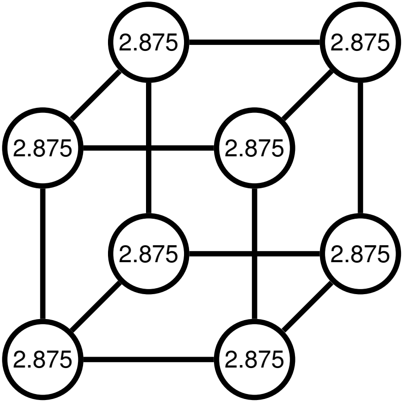

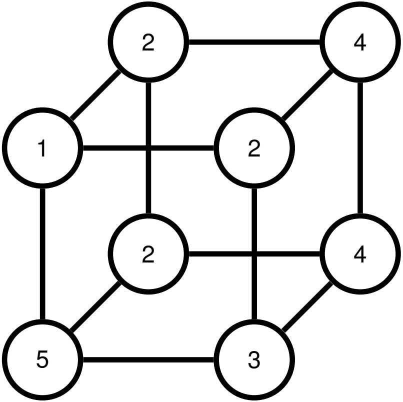

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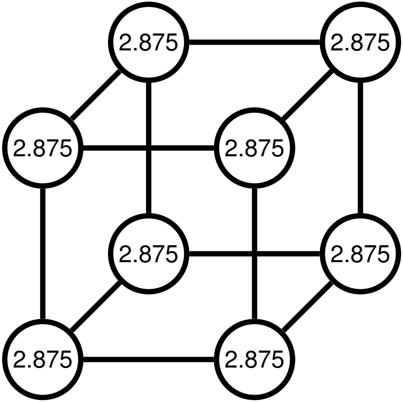

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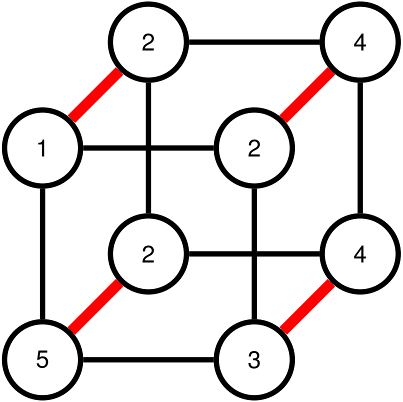

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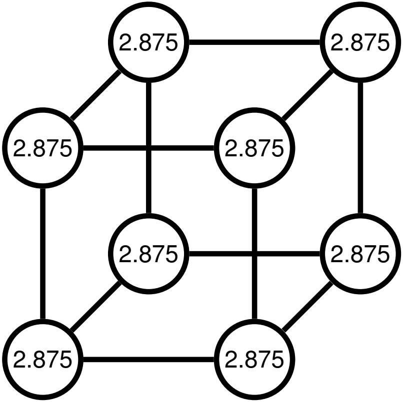

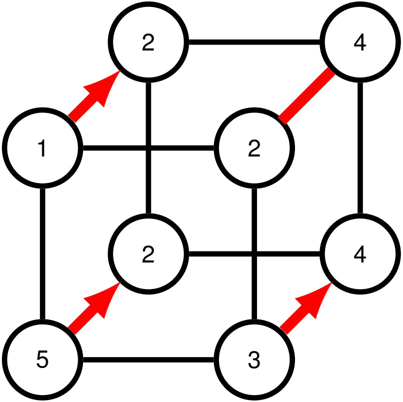

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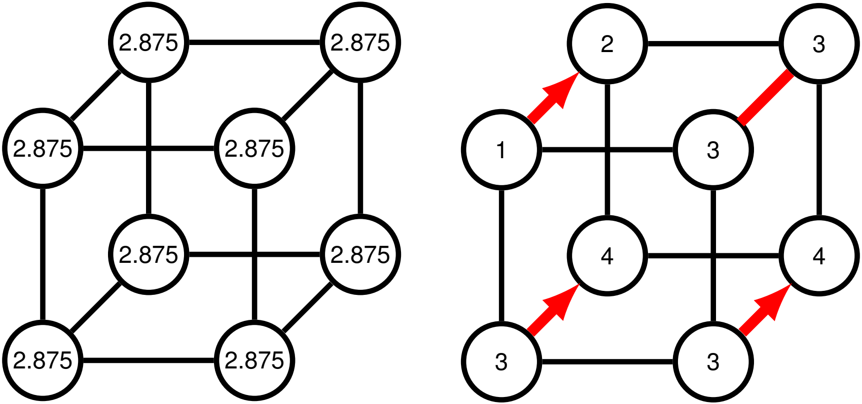

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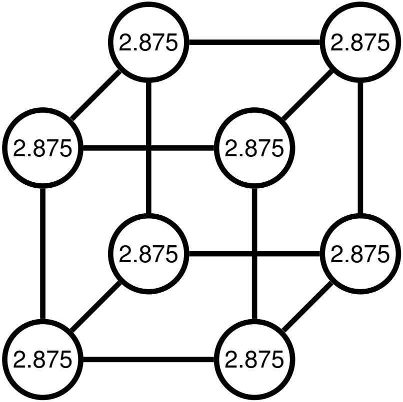

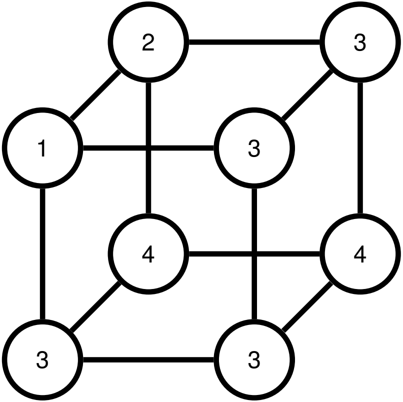

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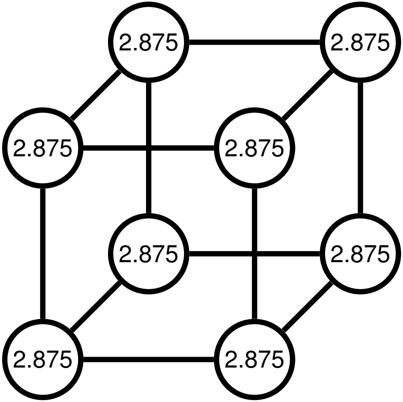

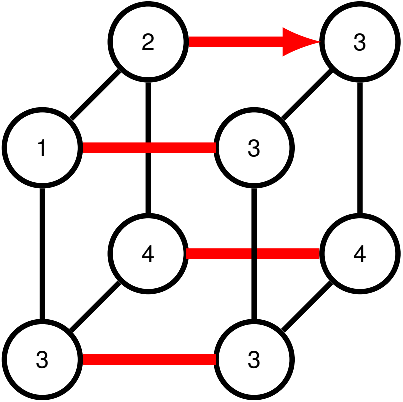

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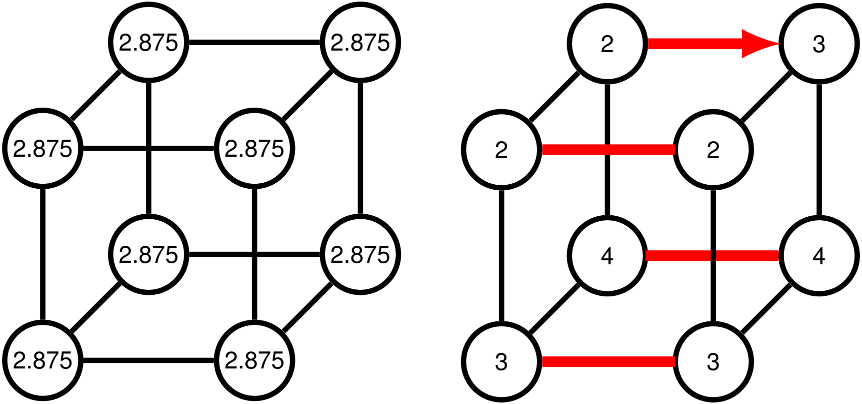

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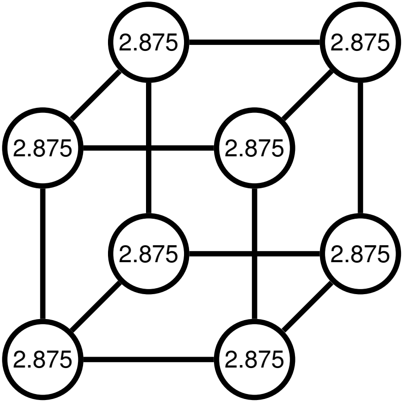

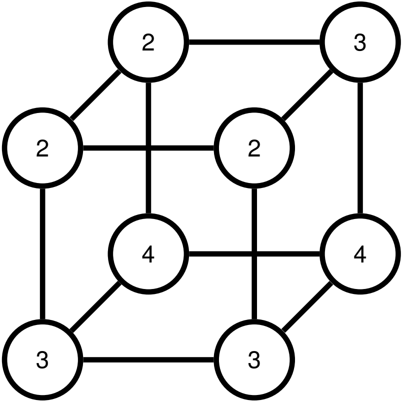

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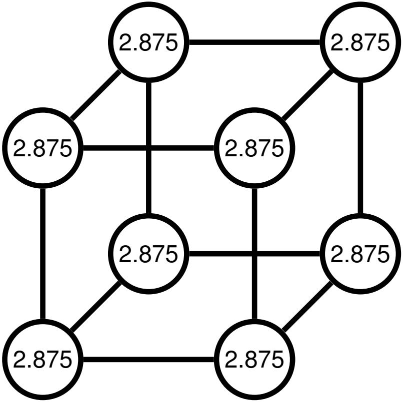

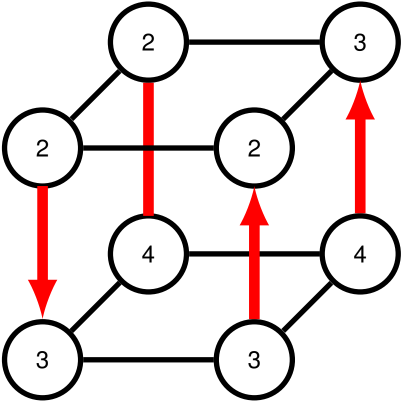

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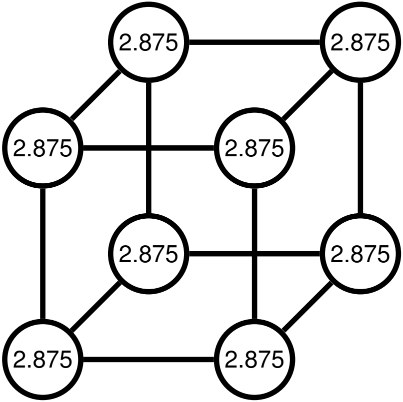

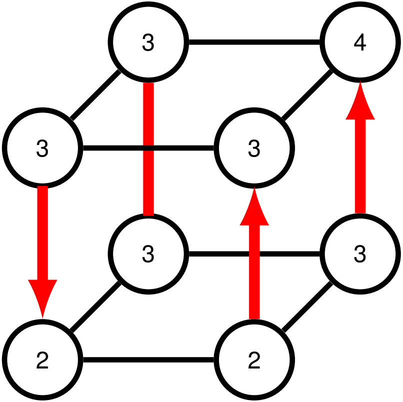

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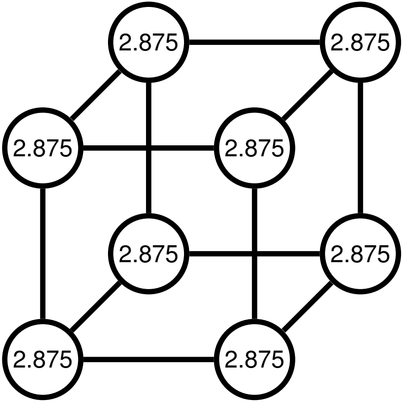

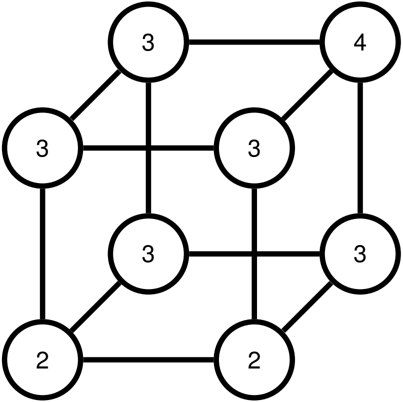

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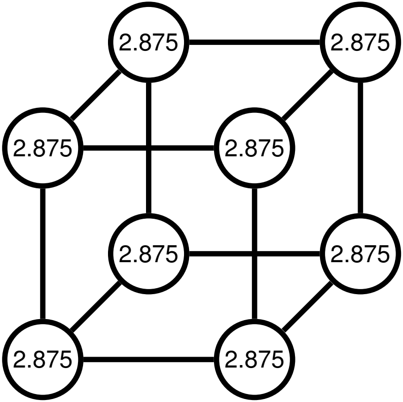

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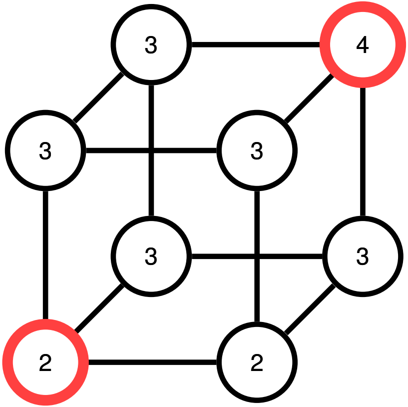
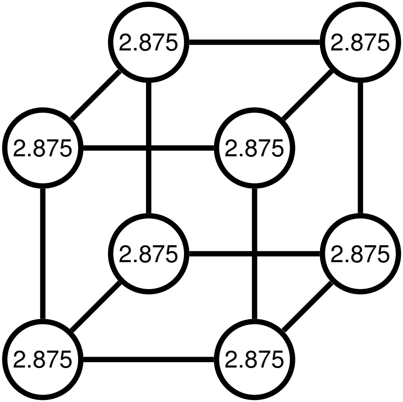


discrepancy of 2

Continuous vs. Discrete Load Balancing



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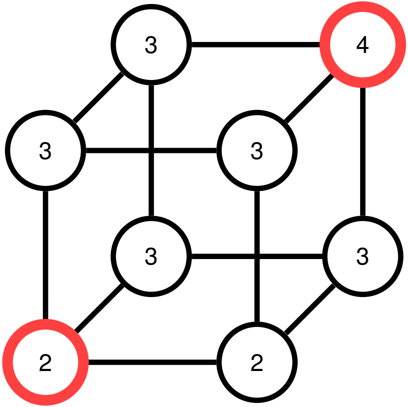


discrepancy of 2

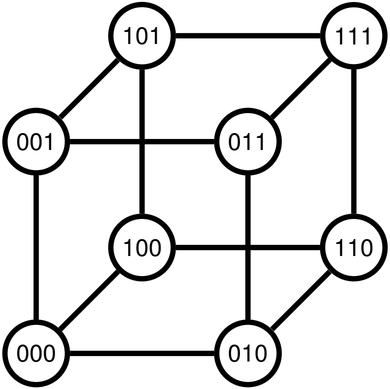
Question

How to minimize the gap between the discrete and continuous case?

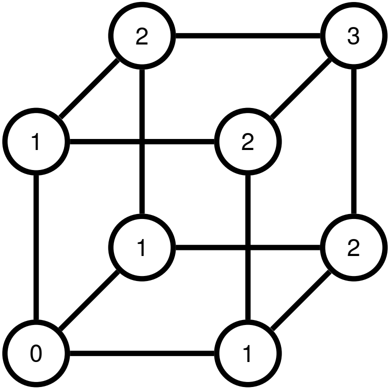
Asynchronous Execution (Smoothing Networks)

Stop

Maximum Discrepancy



Maximum Discrepancy



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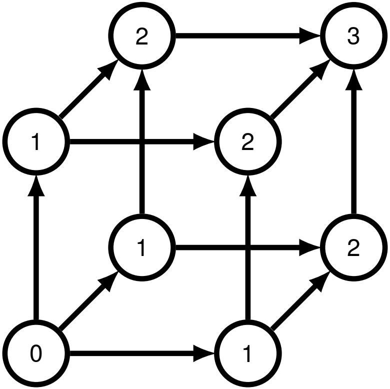
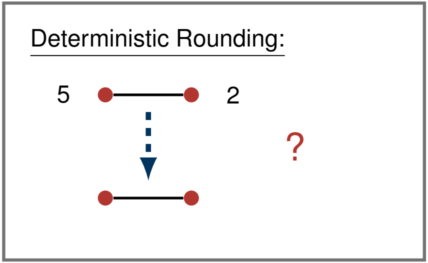


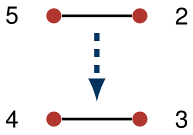
- load vector is never changed
- discrepancy remains 3 (or more generally, $d = \log_2 n$)

Deterministic vs. Randomized Rounding



Deterministic vs. Randomized Rounding

Deterministic Rounding:



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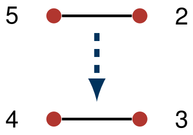

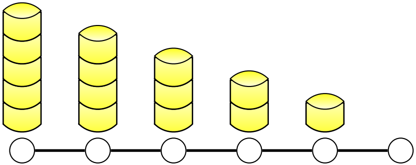

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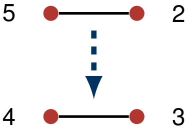
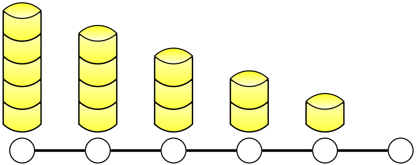
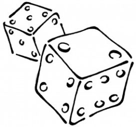
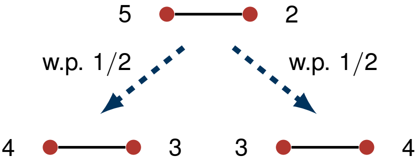




Randomized Rounding:





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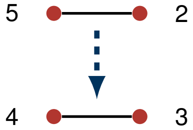
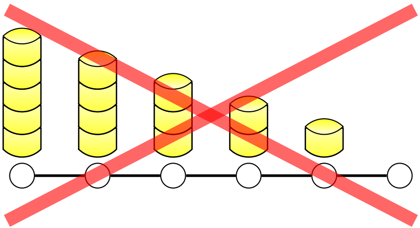
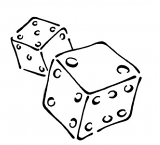
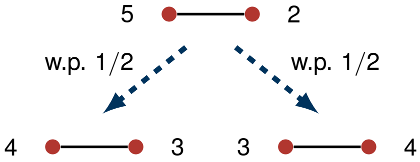




Randomized Rounding:





Upper Bounds for the Hypercube

Arbitrary Rounding (Herlihy, Tirthapura, 2006)

For any initial load vector, the disc. is at most $\log_2 n$ after $\log_2 n$ rounds.

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Randomized Rounding (Mavronicolas, S., 2010)

For any initial load vector, the discrepancy is at most $\log_2 \log_2 n + 4$.

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Randomized Rounding (Mavronicolas, S., 2010)

For any initial load vector, the discrepancy is at most $\log_2 \log_2 n + 4$.

- initial load distribution completely arbitrary (but chosen oblivious to the randomized rounding)
- results hold with probability at least $1 - n^{-1}$

Step 1: Expressing the Rounding Error

$$x_u^t = \frac{x_u^{t-1} + x_v^{t-1}}{2}$$

$$x_v^t = \frac{x_u^{t-1} + x_v^{t-1}}{2}$$

Step 1: Expressing the Rounding Error

$$x_u^t = \frac{x_u^{t-1} + x_v^{t-1}}{2} + e_{u,v}^t$$

$$x_v^t = \frac{x_u^{t-1} + x_v^{t-1}}{2} - e_{u,v}^t,$$

with $e_{u,v}^t$ being the rounding error,

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$$e_{u,v}^t = \mathrm{Odd}(x_u^{t-1} + x_v^{t-1}) \cdot \Phi_{u,v}^t,$$

where the $\Phi_{u,v}^t \in \{-1/2, +1/2\}$ is the (random) orientation.

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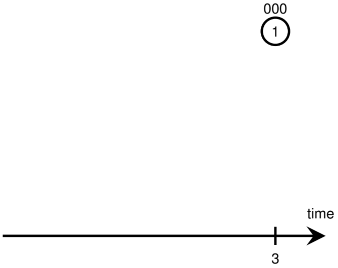
where the $\Phi_{u,v}^t \in \{-1/2, +1/2\}$ is the (random) orientation.

$$e_{u,v}^t \in \{-1/2, 0, 1/2\} \text{ and } \mathbf{E}\left[\, e_{u,v}^t \,\right] = 0.$$

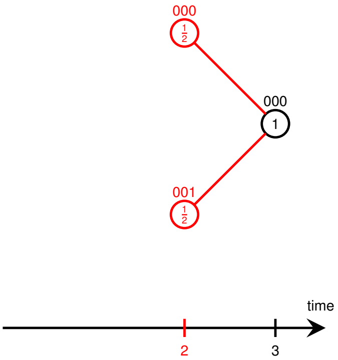
Step 2: Solving and Analyzing the Recursion

x_{000}^{3}



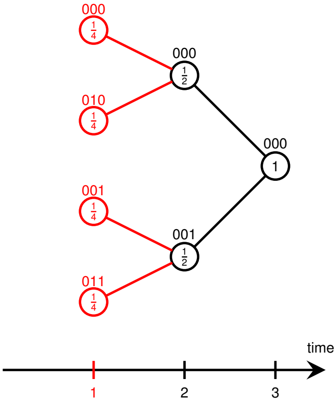
Step 2: Solving and Analyzing the Recursion

$$x^3_{000} = \frac{1}{2}x^2_{000} + \frac{1}{2}x^2_{001} + e^3_{000}$$



Step 2: Solving and Analyzing the Recursion

$$
\begin{aligned}
x_{000}^{3} &= \frac{1}{2}x_{000}^{2} + \frac{1}{2}x_{001}^{2} + e_{000}^{3} \\
&= \frac{1}{4}x_{000}^{1} + \frac{1}{4}x_{001}^{1} + \frac{1}{4}x_{001}^{1} + \frac{1}{4}x_{011}^{1} + e_{000}^{3} + \frac{1}{2}e_{000}^{2} + \frac{1}{2}e_{001}^{2}
\end{aligned}
$$



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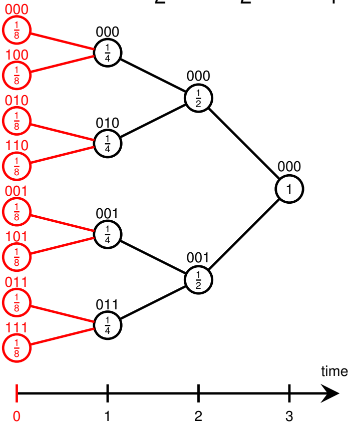
$$
\begin{aligned}
x^3_{000} &= \frac{1}{4}x^1_{000} + \frac{1}{4}x^1_{001} + \frac{1}{4}x^1_{001} + \frac{1}{4}x^1_{011} + e^3_{000} + \frac{1}{2}e^2_{000} + \frac{1}{2}e^2_{001} \\
&= \frac{1}{8}x^0_{000} + \frac{1}{8}x^0_{100} + \frac{1}{8}x^0_{010} + \frac{1}{8}x^0_{110} + \frac{1}{8}x^0_{001} + \frac{1}{8}x^0_{101} + \frac{1}{8}x^0_{011} + \frac{1}{8}x^0_{111} \\
&\quad + e^3_{000} + \frac{1}{2}e^2_{000} + \frac{1}{2}e^2_{001} + \frac{1}{4}e^1_{000} + \frac{1}{4}e^1_{010} + \frac{1}{4}e^1_{001} + \frac{1}{4}e^1_{011}
\end{aligned}
$$



Step 2: Solving and Analyzing the Recursion

$$x^3_{000} = \frac{1}{8}x^0_{000} + \frac{1}{8}x^0_{100} + \frac{1}{8}x^0_{010} + \frac{1}{8}x^0_{110} + \frac{1}{8}x^0_{001} + \frac{1}{8}x^0_{101} + \frac{1}{8}x^0_{011} + \frac{1}{8}x^0_{111}$$
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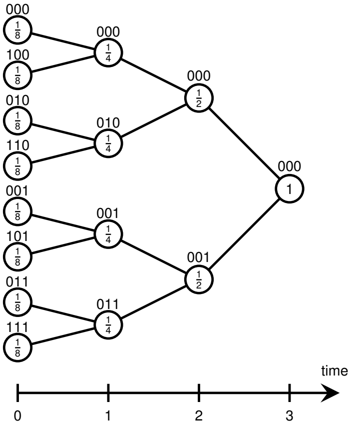

- continuous part and discrete part

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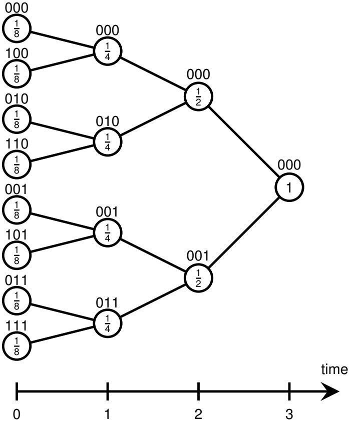

- continuous part and discrete part
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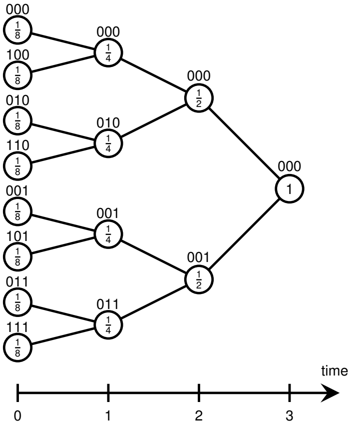

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⇒ loads are divisible, then perfectly balanced

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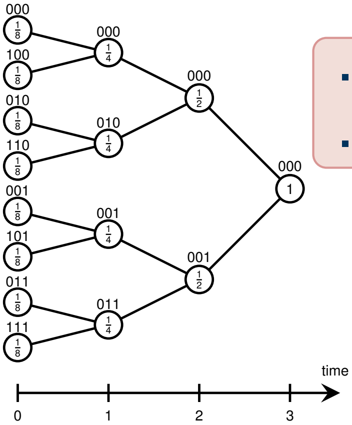

- blue part essentially sum of independent random variables
- ranges decrease exponentially!

- continuous part and discrete part
- continuous part equals the average load

$\Rightarrow$ loads are divisible, then perfectly balanced

Lower Bound for the Hypercube

Upper Bound (Mavronicolas, S., 2010)

For any initial load vector, the discrepancy is at most $\log_2 \log_2 n + 4$.

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Proof Idea of the Lower Bound

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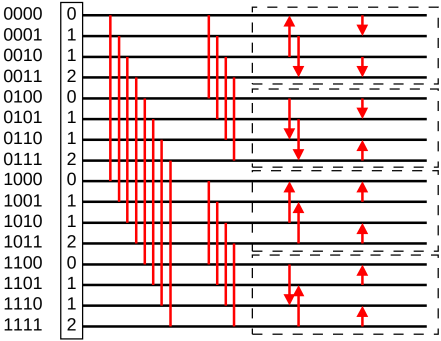

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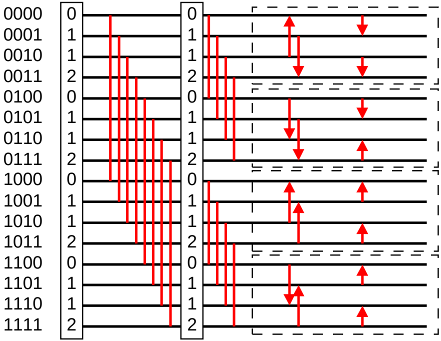

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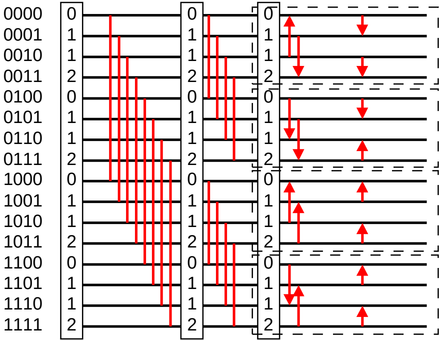

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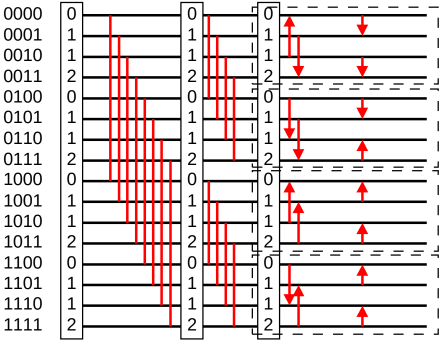

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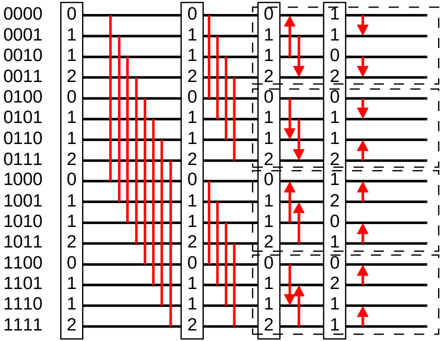

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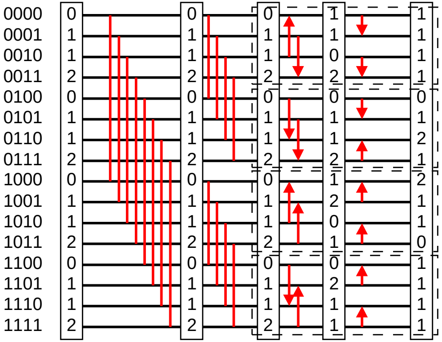

Improving the Lower Bound

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How can we reduce the discrepancy further?

Average Case Input (Friedrich, S., Vilenchik, 2011)

Even if the initial load at a node is chosen i.u.r. in $\{0, 1, \ldots, n-1\}$, then the discrepancy is at least $\frac{1}{2} \cdot \log_2 \log_2 n - 2$ w.p. $1 - n^{-1}$.

New Attempt

Old Protocol (Mavronicolas, S., 2010)

- rounds $0, 1, \ldots, \log_2 n - 1$
- in round i communicate along dimension i
- discrepancy $\leqslant \log_2 \log_2 n + 4$

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- rounds $0, 1, \dots, 3 \log_2 n - 1$
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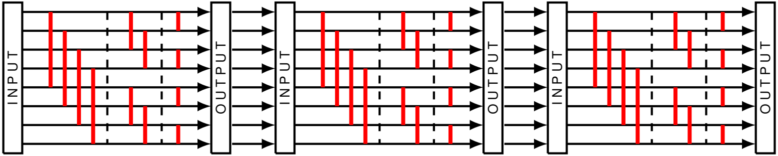
- discrepancy $\leqslant 2$

Analysis



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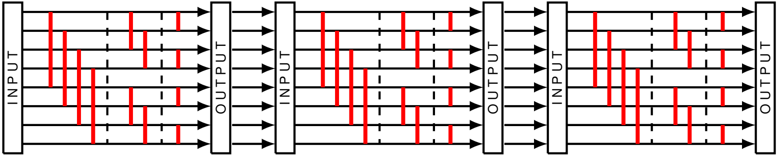

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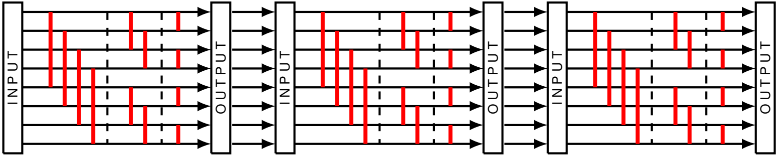

- after the first $\log_2 n$ rounds, discrepancy is at most $\log\log_2 n + 4$
- analysis consists of $\log_2 \log_2 n + 2$ phases: each phase reduces D by 1
- focus on the last phase (D drops from 3 to 2)

Maximum-Paths

- idea: keep track of the maxima in the load vector

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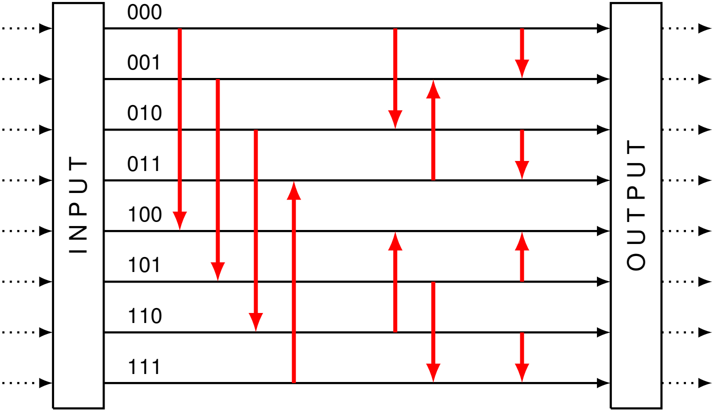

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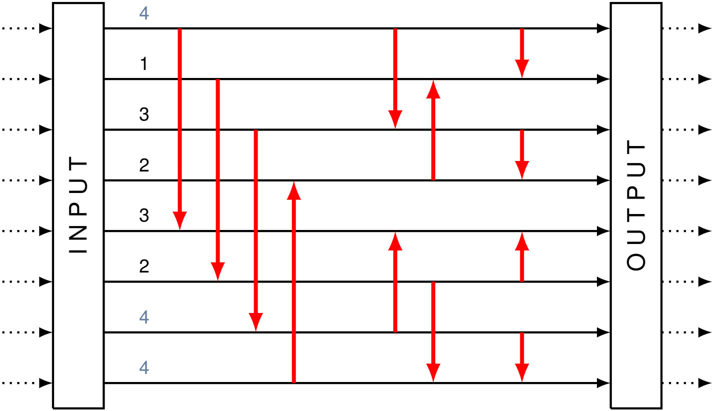

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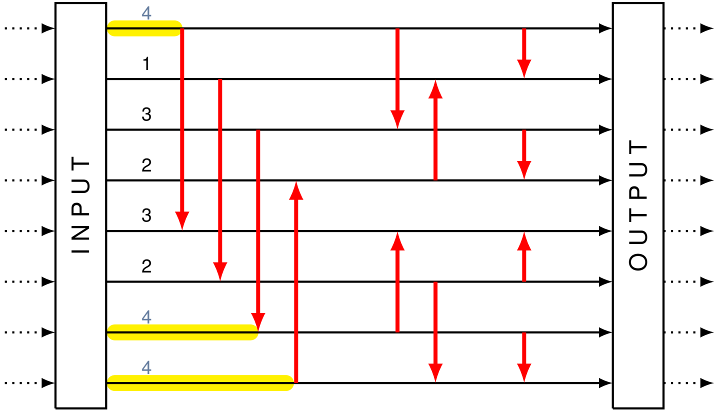

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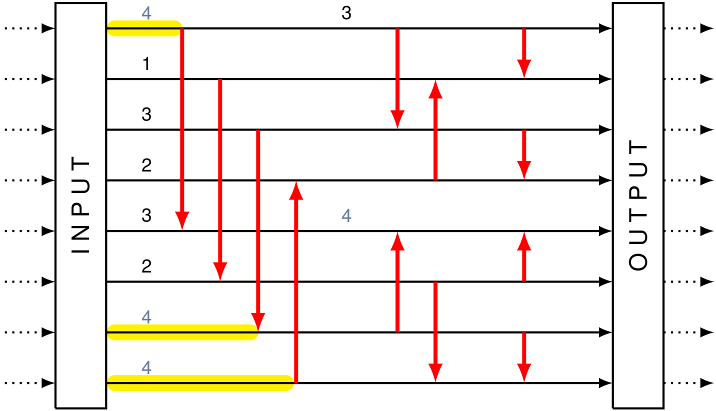

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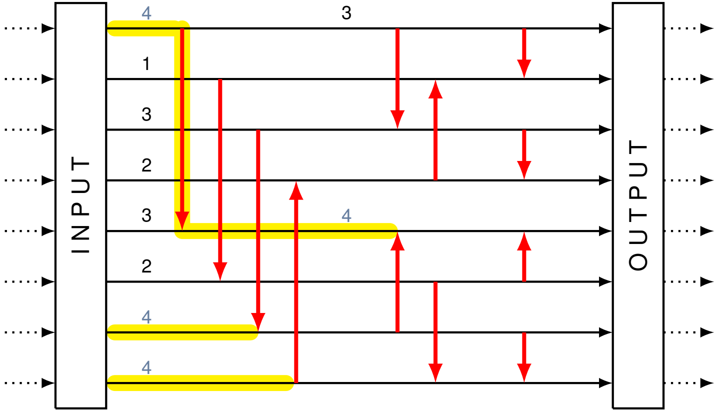

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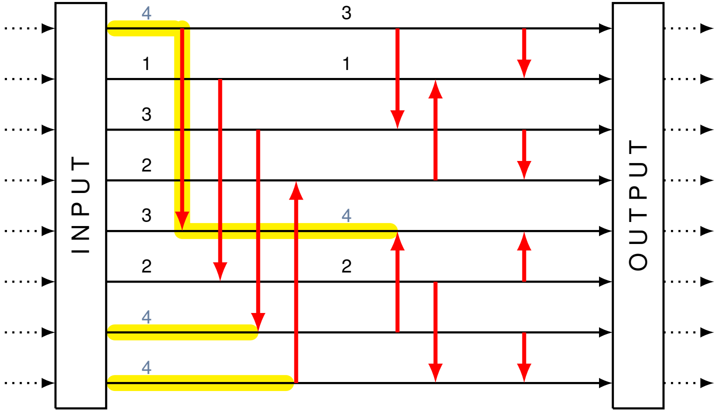

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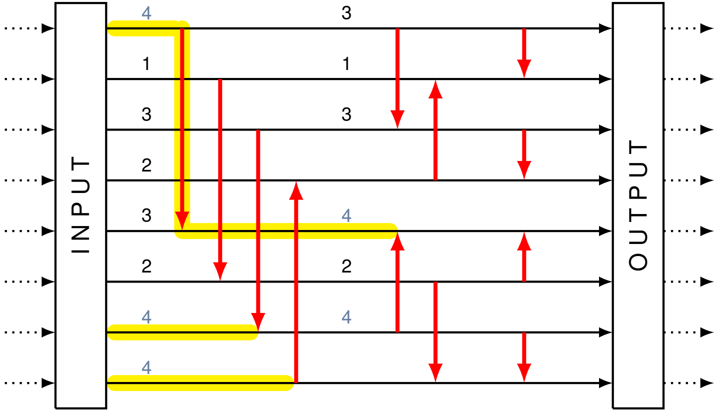

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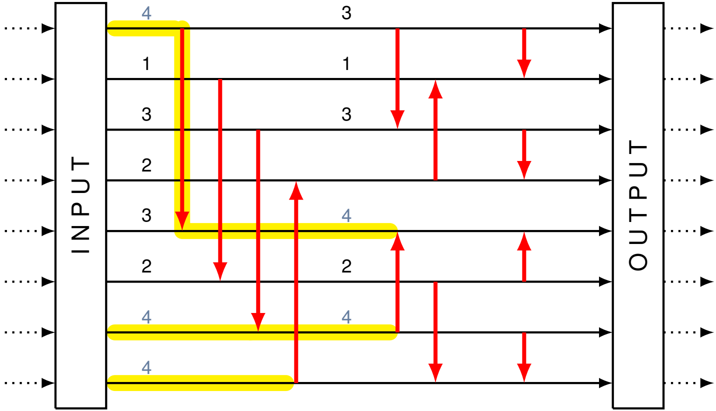

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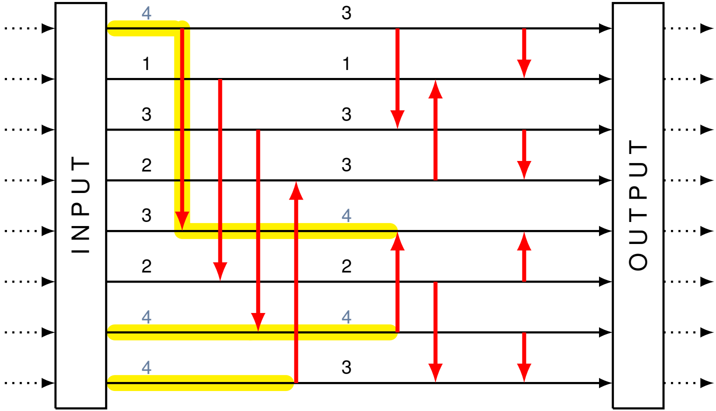

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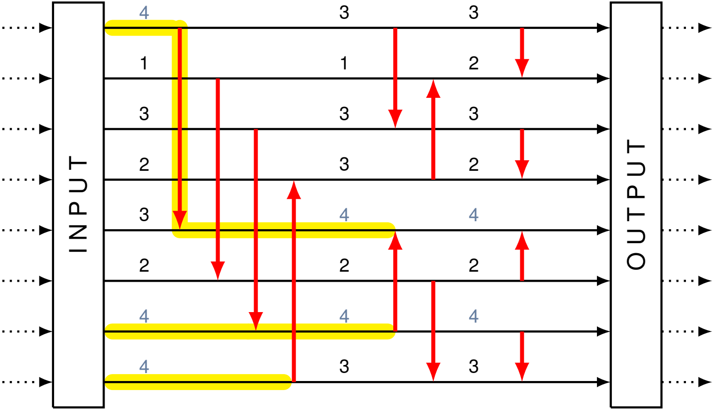

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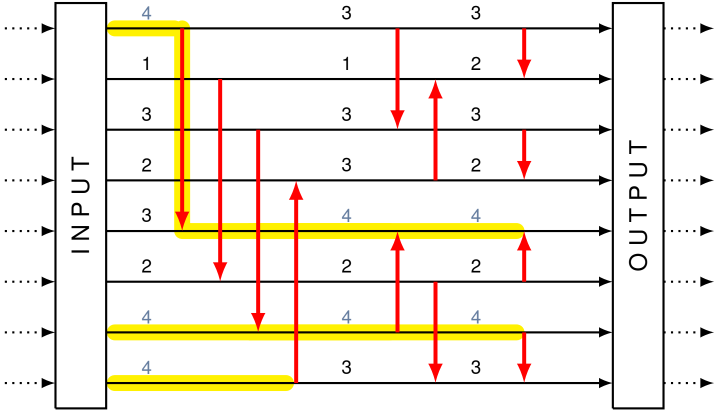

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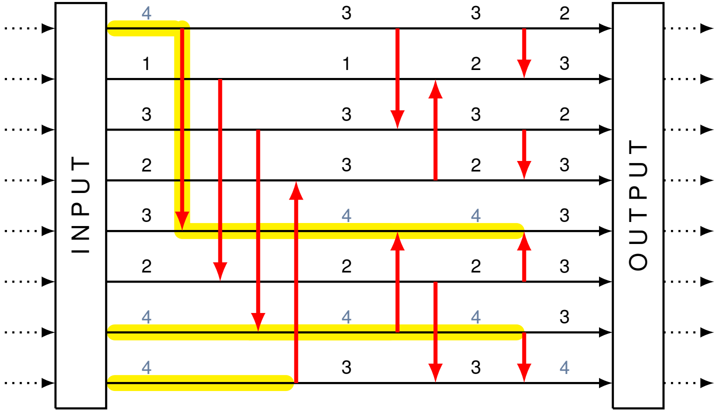

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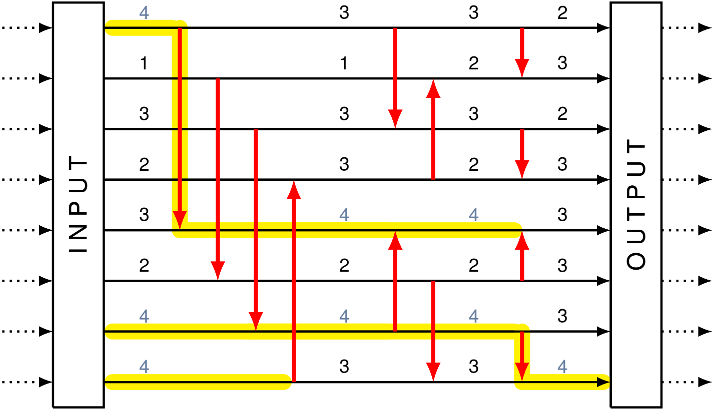
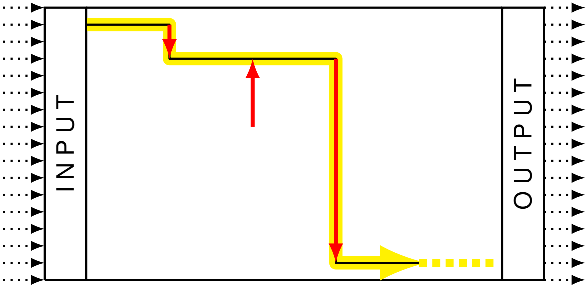


Reducing Discrepancy to 2



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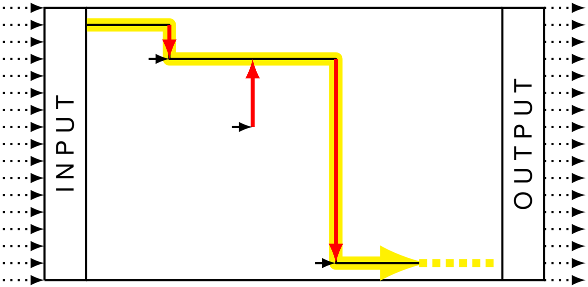

- maximum path survives $\Leftrightarrow$ all inputs $\geqslant$ max -1

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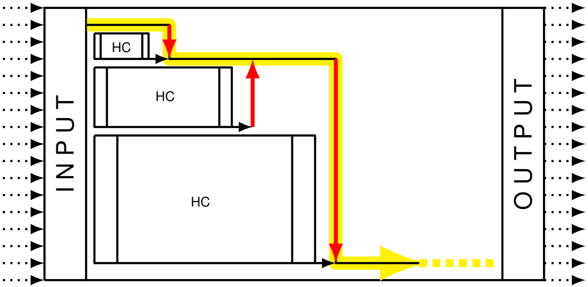

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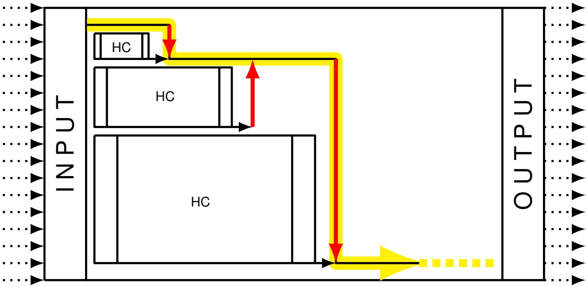

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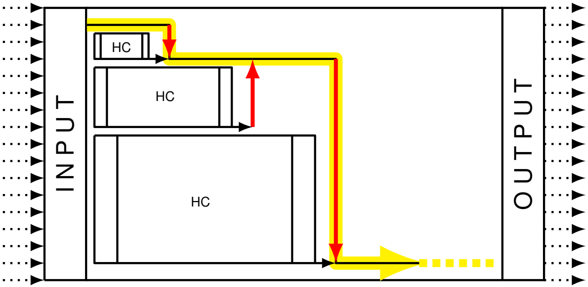

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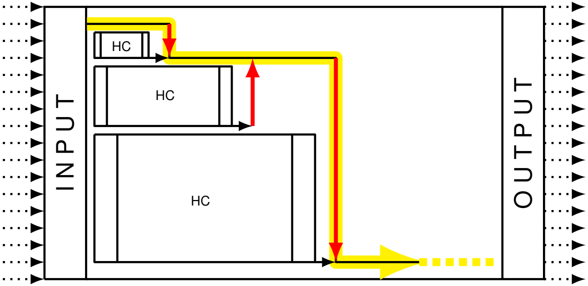

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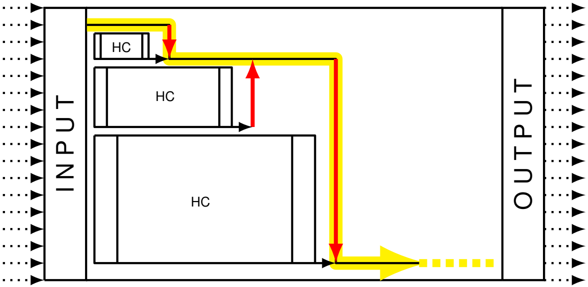

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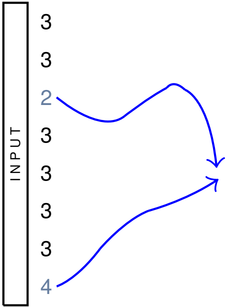
$\Rightarrow$ with prob. $1 - n^{-1}$, discrepancy is 2 $\square$

Discrepancy of 1?

INPUT	3
	3
	2
	3
	3
	3
	3
	4

Discrepancy of 1?



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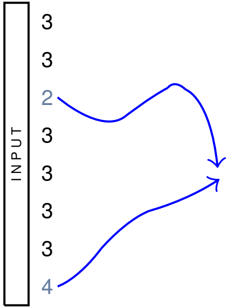

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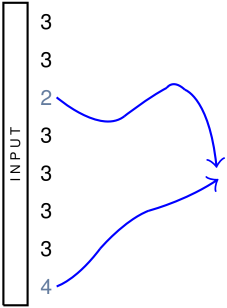

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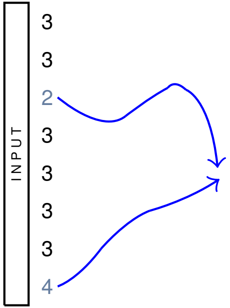

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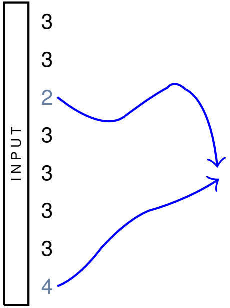

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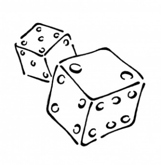
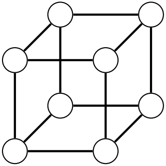
(holds for any network and any matchings)

Hypercube Summary

Randomized Rounding

- $D = \log_2 \log_2 n \pm \Theta(1)$ after $\log_2 n$ rounds
- $D = 3$ after $\log_2 n + o(\log n)$ rounds
- $D = 2$ after $3 \log_2 n$ rounds
- $D = 1$ not possible in $o(n)$ rounds





Hypercube Summary

Randomized Rounding

- $D = \log_2 \log_2 n \pm \Theta(1)$ after $\log_2 n$ rounds
- $D = 3$ after $\log_2 n + o(\log n)$ rounds
- $D = 2$ after $3 \log_2 n$ rounds
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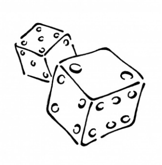
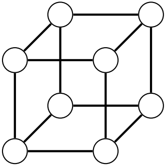
$\Rightarrow$ since $\log_2 n$ rounds are necessary, hypercube is very close to the optimal network





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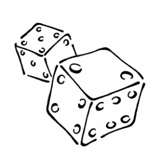

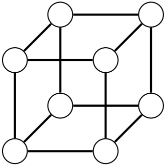

What about general graphs?

Outline

Introduction

Load Balancing on Hypercubes

Load Balancing on Arbitrary Graphs

Conclusions

Load Balancing: Results

- $1-\lambda$: spectral expansion of the n-vertex regular graph

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For any graph, the discrepancy is $\mathcal{O}\left(\frac{\log n}{1-\lambda}\right)$ after $\mathcal{O}\left(\frac{\log(Kn)}{1-\lambda}\right)$ rounds.

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This is at least the diameter of the graph!

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No diff. between continuous and discrete case

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Analyzing Recursion

x_u^t

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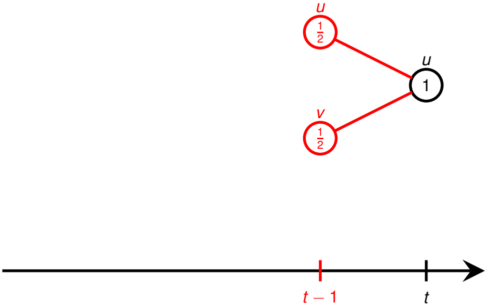

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$$
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\end{aligned}
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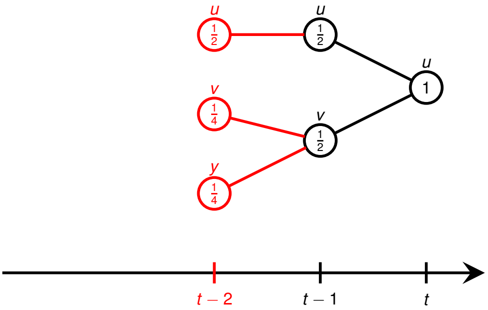

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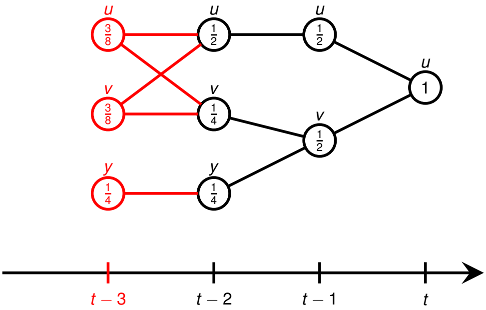

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&\quad + e_{u,v}^{t-1} + \frac{1}{2}e_{v,y}^{t-2} + \frac{1}{2}e_{u,v}^{t-3} - \frac{1}{4}e_{u,v}^{t-3} + \frac{3}{8}e_{r,u}^{t-4} + \frac{1}{4}e_{y,z}^{t-4}
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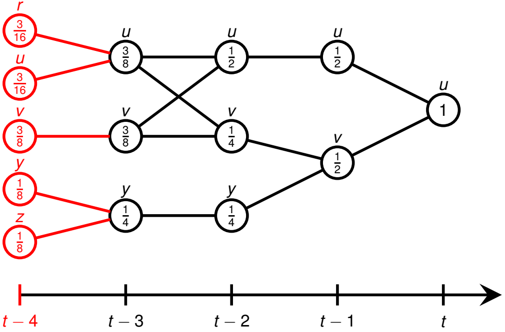

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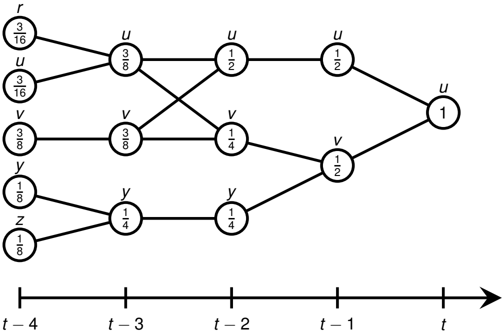

- continuous part and deviation

Analyzing Recursion

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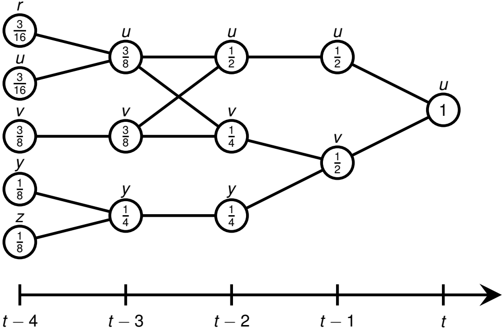

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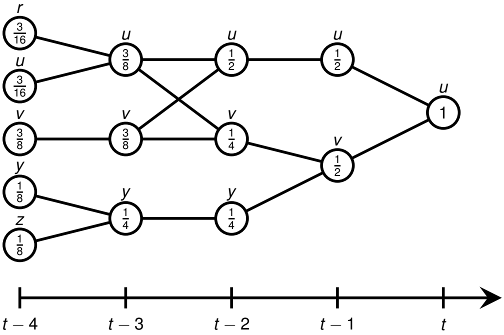

- continuous part and deviation
- coefficients of x^t converge
- coefficients of e's keep track of this convergence

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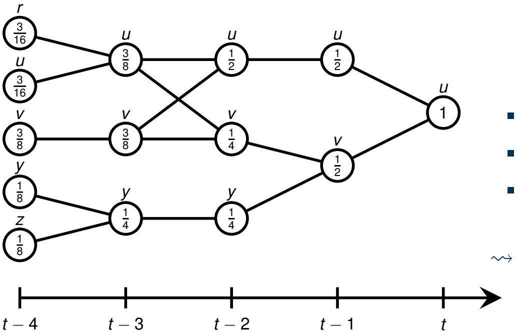

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⇝ deviation part dominated by $\mathcal{N}(0,2)$

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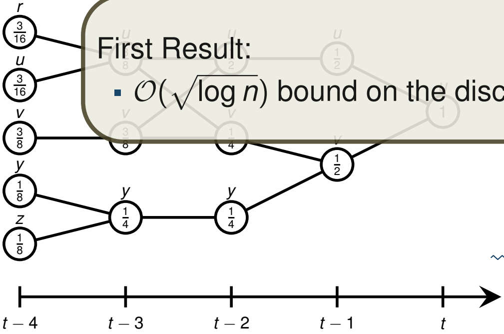


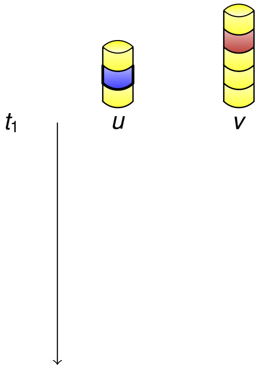
First Result:

- $\mathcal{O}(\sqrt{\log n})$ bound on the discrepancy

- continuous part and deviation
- coefficients of x^t converge
- coefficients of e's keep track of this convergence

$\rightsquigarrow$ deviation part dominated by $\mathcal{N}(0, 2)$

Step 1: Token Movements as Random Walks



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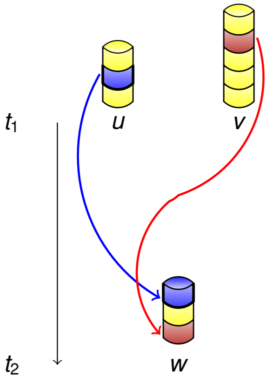

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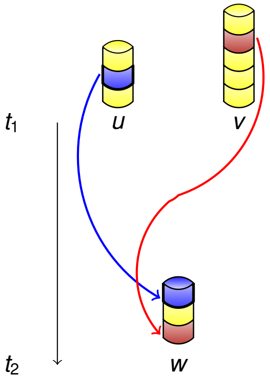

Negative Correlation

- **Pr** [blue token on w] $= \mathbf{M}_{u,w}^{[t_1,t_2]}$, **Pr** [red token on w] $= \mathbf{M}_{v,w}^{[t_1,t_2]}$

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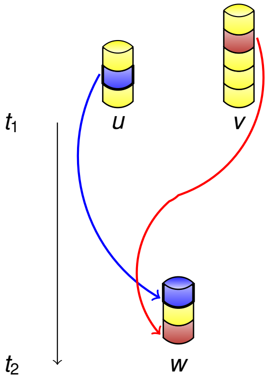

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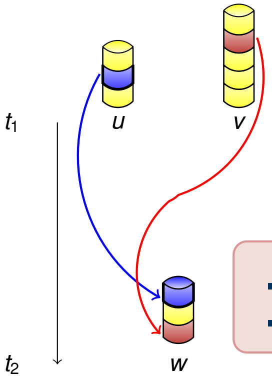

- Allows use of Chernoff bounds for independent r.v.'s
- Fewer tokens yields better concentration

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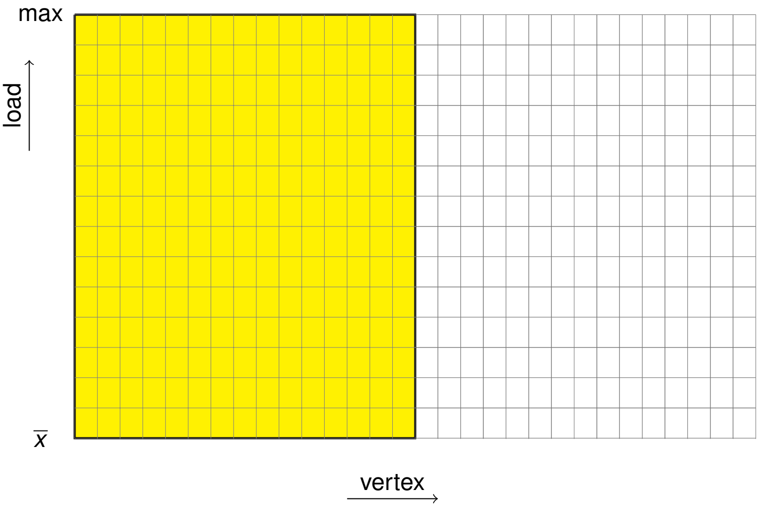
Step 2: Sparsification of the Load Vector



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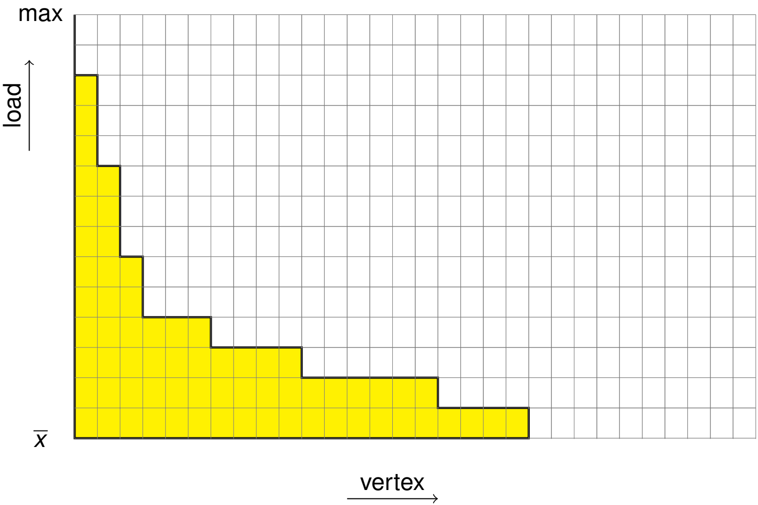

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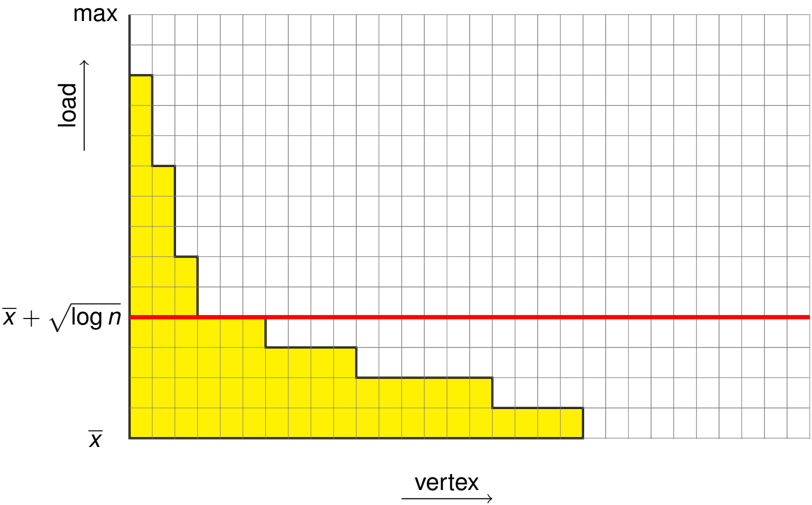

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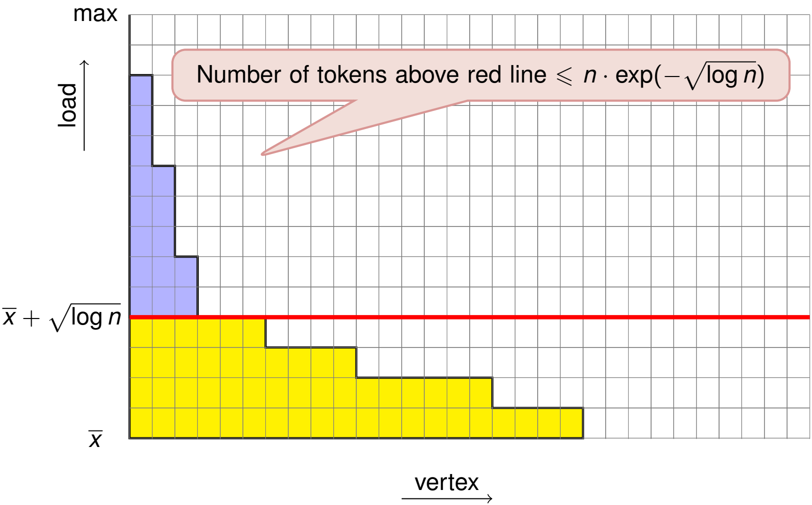

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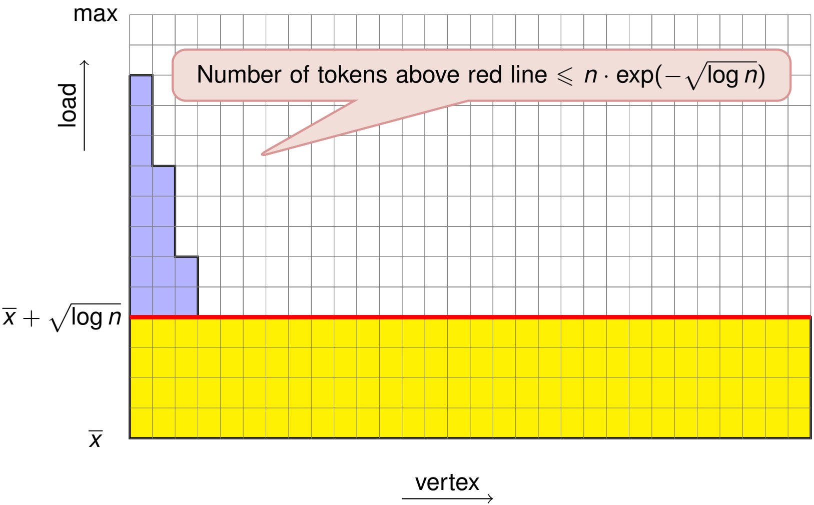

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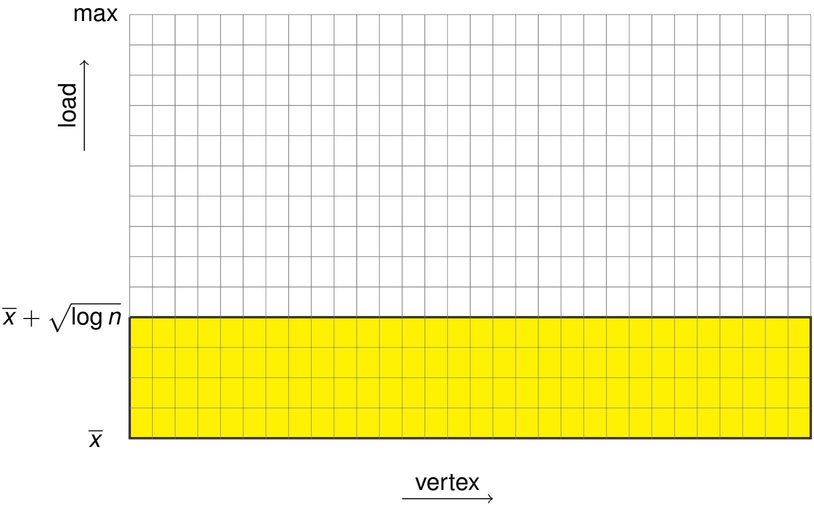

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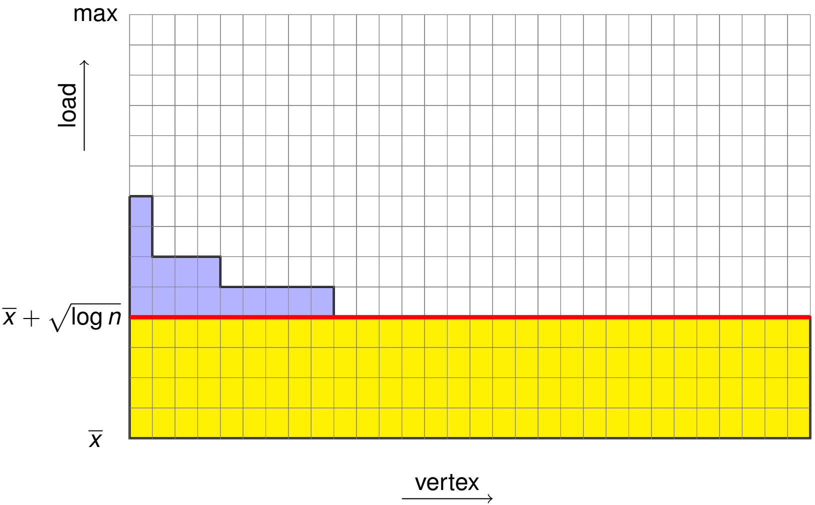

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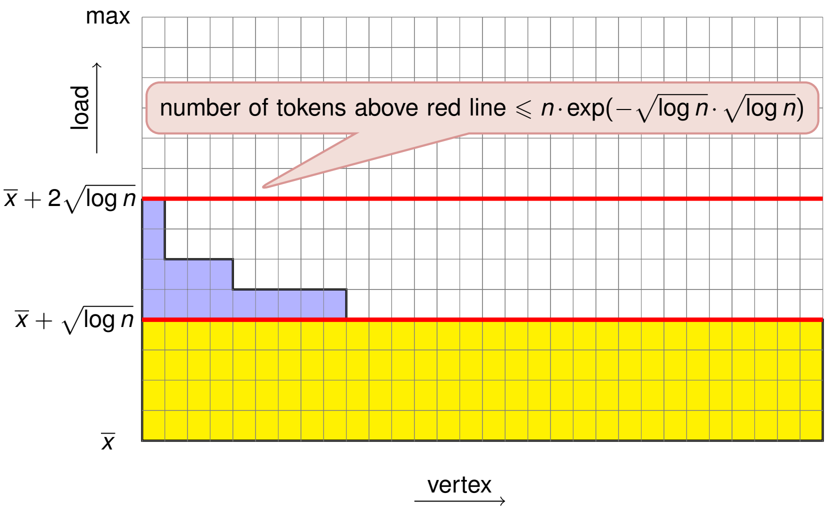
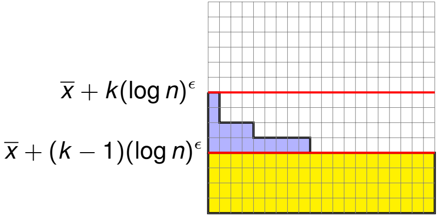


Step 2: Sparsification of the Load Vector

- Let $\epsilon > 0$ be any value
- After k iterations, number of blue tokens is $\leqslant n \cdot \exp(-(\log n)^{\epsilon \cdot k})$.



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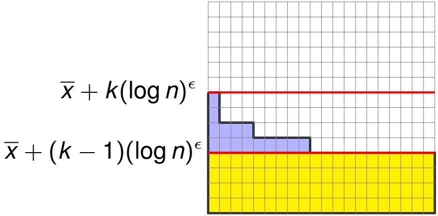

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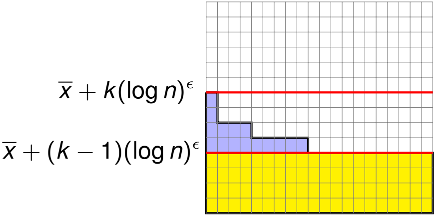
- lower bound on minimum load by symmetry



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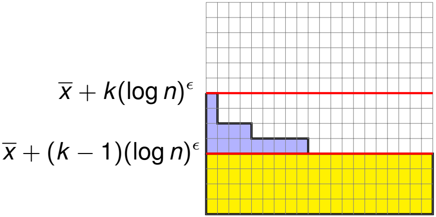

Preliminary Results (also for non-regular graphs)

- After $\mathcal{O}\left(\frac{1}{\epsilon} \cdot \frac{\log(Kn)}{1-\lambda}\right)$ rounds, discrepancy is $\mathcal{O}((\log n)^{\epsilon})$.

Step 2: Sparsification of the Load Vector

- Let $\epsilon > 0$ be any value
- After k iterations, number of blue tokens is $\leqslant n \cdot \exp(-(\log n)^{\epsilon \cdot k})$.
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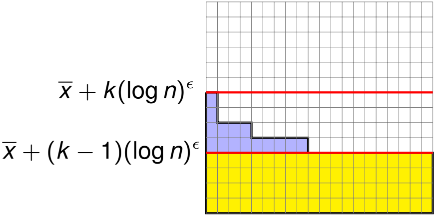

Preliminary Results (also for non-regular graphs)

- After $\mathcal{O}\left(\frac{1}{\epsilon} \cdot \frac{\log(Kn)}{1-\lambda}\right)$ rounds, discrepancy is $\mathcal{O}((\log n)^{\epsilon})$.
- After $\mathcal{O}\left(\log\log n \cdot \frac{\log(Kn)}{1-\lambda}\right)$ rounds, discrepancy is $\mathcal{O}(\log\log n)$.

Step 2: Sparsification of the Load Vector

- Let $\epsilon > 0$ be any value
- After k iterations, number of blue tokens is $\leqslant n \cdot \exp(-(\log n)^{\epsilon \cdot k})$.
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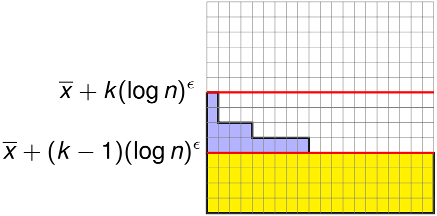

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(Much) more work required to get constant discrepancy...

Outline

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Load Balancing on Arbitrary Graphs

Conclusions

Hypercube

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- Discrepancy of 3 after $\log_2 n + o(\log n)$ rounds
- Discrepancy of 2 after $3\log_2 n$ rounds

Conclusion

- Very good understanding
- Since $\log_2 n$ rounds are necessary, hypercube is "optimal network"
- Proofs: Chernoff bounds using independence

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Arbitrary Graphs

Results for Random Matchings

- Constant Discrepancy in $\mathcal{O}\left(\frac{\log(Kn)}{1-\lambda}\right)$ rounds for any regular graph

Techniques

- Movements of Tokens instead of rounding errors
- Sparsification: Reduce general problem to sparse vectors

Future Work

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Thank you for your attention!