Sublinear communication in a message-passing network: Broadcast and MST

Valerie King
University of Victoria,
Vancouver Island, BC
Canada
Network with $n$ nodes, $m$ edges
A node wants to send a message to all other nodes. How many messages are needed?
Use **flooding** from a source node $O(m)$ messages.
Can we do better than $O(m)$ messages?
Given a spanning tree, can send broadcast in \( n-1 \) messages.
Given a spanning tree, can send broadcast in $n-1$ messages. What if each node has only local info?
Talk Outline

• Models and lower bounds for spanning tree (and MST) construction.
• Upper bounds
  – Synchronous
  – Asynchronous
• An even simpler communication problem
• Time-Communication Tradeoffs
• Open problems
Nodes have distinct IDs

Know only **LOCAL** info re topology
Auerbuch, Goldreich, Peleg, Vainish (AGPV) (JACM 1990)

--$\text{KT}_0$ each node has a port to each neighbor

--$\text{KT}_1$ each node knows its neighbors’ IDs
communication

• Synchronous: messages sent in a round received in the same round.
• Asynchronous: algorithm may wake up all or some nodes to start; later messages are sent in response to receipt of a message. ("event-driven")
A network builds a subgraph $G'$ when nodes mark their incident edges which are in $G'$. 

![Diagram of network with nodes A, D, C, H, J, G, E, and F connected]
Can we do better than flooding?
“No, according to folklore.” (AGPV)

“If each proc knows only its ID and the IDs of its neighbors, then flooding (O(m)) is the best that can be done [even in the synchronous model]”

“Obvious” for $KT_0$ (AGPV1990) ???
lower bounds for spanning tree: $KT_0$

Kutten, Pandurangan, Peleg, Robinson, Trehan (PODC2013, JACM 2015)

Constructing a tree requires $\Omega(m)$ communication (where range of IDs has size $n^4$ )

Even if

• synchronous
• nodes are all initialized at the start
• alg is Monte Carlo
• nodes know $n$
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First even for deterministic bound!

But much earlier lower bound for MST: Korach, Moran, Zaks PODC 1984)
Constructing a tree requires $\Omega(m)$ communication where range of IDs has size $2n$.

Even if
- synchronous
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Provided...
lower bounds for spanning tree in $\text{KT}_1$

AGPV 1990

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Provided each message contains a constant number of ids and ids can only be compared
More on $KT_1$ : Without proviso

- AGPV requires use of Ramsey Theory, very large ID space, so that IDs are not $O(\log n)$ bits.

--->The lower bound is wide open in $KT_1$ even for deterministic algorithms.
An aside: a non-communications network lower bound

• Where knowledge of an arbitrary graph’s edges can be partitioned arbitrarily among n parties in a clique, $\Omega(n^2)$ bits of communication are required. Phillips, Verbin, Zhang (SODA 12)
Upper bounds
Main Idea: How to find an edge leaving a marked tree $T$ ("outgoing edge") with $\tilde{O}(|T|)$ bits.
Old idea:
The sum of the degrees of nodes in a component is:

- **Even**  *if it has no outgoing edges*—*each edge counted twice*

- **Odd**  *if exactly one edge is “leaving” component*—*one edge counted once*
FindAny: How to find an edge leaving the tree \( T \) if there is exactly one such edge?
EDGE name \(<a,b>\): a (in binary) followed by b (in binary), if \(a < b\)

For each vertex a form vector \(v(a)\)

\[ v(a) = \text{XOR (edges incident to a)} \]
XOR of \( v(1) = 001010 \)
in \( S \)
\(+ v(2) = 001000 \)
\(+ v(7) = 010111 \)

= \( 010101 \) = name of edge leaving \( S \) when there is exactly 1 edge in cutset
To reduce cutset to size 1:

Use 2-wise independent hash to sample edges

\[ h: \text{[edge names]} \rightarrow [0, 1, \ldots, n^2] \]

Edge \( e \in \text{Sample } i \) iff \( h(e) \leq 2^i \)

\[ i = 0, 1, 2, \ldots, 2 \lg n \]

\( \rightarrow \) w/ constant prob, for \( i = \lg |\text{cutset}| \), Sample \( i \) contains exactly 1 edge in the cutset.
Basic communication pattern: broadcast-and-echo over tree edges

“Leader” of tree broadcasts message down tree,
Basic communication pattern:

broadcast-and-echo over tree edges

“Leader” of tree broadcasts message down tree,
Response composed from leaves back up to leader
FindAny Alg in 3 broadcasts-and-echoes: First broadcast

“Leader” of tree picks hash h and broadcasts it
**FindAny Alg in 3 broadcasts-and-echoes:**

first echo

Each node \(v\) computes word \(b(v)\) where \(i^{th}\) bit \(b_i(v)\) = parity of \(v\)'s incident edges\) in Sample \(i\)

Nodes in tree echo up to compute \(B = \text{XOR}_{v \in T} b(v)\)

Leader computes \(\min = \text{minimum } i \text{ s.t.} \ i^{th} \text{ bit } B_i = 1\)
FindAny: finds edge leaving tree $T$ in 3 broadcast-and-echoes (cont’d)

**STEP 2**
Leader broadcasts $\min$
Echo: XOR names of edges $e$ incident to all nodes in $T$ with

$$h(e) \leq 2^{\min} \text{ (i.e., in Sample } \min)$$
All sampled edges which are not outgoing cancel out with XOR, leaving hopefully one edge

**STEP 3.** Broadcast-and-echo $e$: Leader **verifies** that $e$ is name of exactly one edge in the cutset
FindAny does a lot!

Observe:

1. With constant prob, returns an outgoing edge
2. Edge is a random edge in cutset
3. $n^2 / \min$ is an estimate of the cutset size

(Observations 2,3 needed for later)
Sketch of synchronous alg
Boruvka style algorithm to build tree:
Phase: In parallel, w/constant prob each tree finds an edge leaving and merges  (must prevent cycles)
Minimum spanning tree

- Modify FindAny to do binary search on edge weight ranges to find a minimum weight outgoing edge
Doesn’t work for asynchronous

- No global clock, except for initial wake-up, actions are event-driven
- One tree will grow, one node at a time, while the others sit, for a cost of
- $\sum_{i=1}^{n} i = O(n^2)$
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Why doesn’t the Gallagher Humblet Spira (GHS) (1983) technique work?
Asynchronous alg
Ali Mashreghi, K DISC 2018
All nodes awake:

• Low degree (< $\sqrt{n \log n}$) nodes send to all their neighbors
• With prob $1/\sqrt{n}$) nodes choose themselves as stars
• stars send to all their neighbors
Grow tree $T$ in phases, from root:

After each phase:
A. A high degree node (and a new adjacent star node) is added to $T$

OR

B. $T$ is expanded until the number of outgoing edges to low degree neighbors is reduced by a half.

$\sqrt{n}$ stars $\rightarrow \sqrt{n}$ type A phases

There are no more than $\lg n$ type B phases between each type A phase

$\rightarrow \sqrt{n} \lg n$ total phases
How to grow tree $T$ (Step 1):

1) Expand $T$ recursively:

- **Low degree nodes** in $T$ bring in their all neighbors
- **High degree nodes** in $T$ bring in at least one Star node w.h.p and
- **Star nodes** bring in all their neighbors.
How to grow T (step 2)

2). Find an outgoing edge to a High degree node using FindAny

Else { FindAnys fails to find a High degree node}

**WAIT** until T receives messages from Low degree nodes over half the outgoing edges
How do you WAIT asynchronously??

Use **FindAnys** to estimate cutset size \( K \)

For each message sent to \( T \), trigger new phase with prob \( \sim 2/K \).
The next Expand either finds a high degree node or the number of outgoing edges to nodes with low degree is reduced by half.
Analysis

• Cost per phase $O(n \log n)$ words, each of $O(\log n)$-bits and there are $O(\sqrt{n \log n})$ phases
• For a total of $O((n \log n)^{3/2})$ words.
Asynchronous minimum spanning tree

1. Construct a spanning tree
2. Using it, root can synchronize the merging of fragments by level number, as in GHS
Can be done with $\tilde{O}(n \log n)$ messages.
A simpler model for understanding lower bounds?

- One-way communication, a Coordinator
- Each node sends to Coordinator, $X$ bits
- Coordinator outputs spanning tree or connected/not connected
- With public randomness, this can be done with connectivity streaming method with $X = O(\log^3 n)$ bits (Monte Carlo) and this is tight (Nelson, Yu 2018)
Coordinator—Public randomness
=streaming technique

Public randomness is used to specify $\log n$ hash functions.

For each hash function:
Each node sends the XOR of its neighbors’ names which hash to $[1,2^i]$ range for $i=0,\ldots, 2\log n$

For $i=1,2,\ldots, c \log n$, Coordinator uses $i^{th}$ hash function result to find outgoing edges to compute $i^{th}$ tier of Boruvka tree.

To verify whp that XOR is indeed an edge name, extra info must be sent by each node.
(Ahn,Guha,McGregor, SODA 2012 )
Observe: Given a set of $n^c$ elements, we can deterministically encode each element using $O(k \log n)$ bits so that the XOR of any subset of $k$ elements is unique.
Sending to Coordinator:

Let $k = \sim \sqrt{n}$

- Each node sends the $O(\log n)$ bit name of each incident edge with probability $(\log^2 n)/k$

- Each node $v$ encodes each of its incident edges $\{u,v\}$ using $k \log n$ bits and sends the XOR of these encodings, $\text{XOR}(v)$
Coordinator:

1) Uses sampled edges to create connected components.
   Note: remaining components have small cuts w.h.p.

2) For each component C, uses XOR of coded strings from nodes in C to determine all edges between the components
Optimizing for $k$

- This gives $X = \tilde{O}(n^{1/2})$ bits.
- (Holm, K, Thorup, Zwick)
Time-communication tradeoffs

In the synchronous model for MST:

• $\tilde{O}(n)$ bits, $\tilde{O}(\text{diameter}(\text{MST}))$ time
  (Mashreghi, K, ICDCN 2017)

• $\tilde{O}(\min\{n^{3/2},m\})$ bits, $\tilde{O}(\text{diameter}(G)+\sqrt{n})$ time
  Ghaffari, Kuhn DISC 2018
Open problems

1. **Lower bound** even for det algs for broadcast tree in general $KT_1$ model

   a) Simpler: In the one-way communication model, anything better than $\Omega(\log^3 n)$ for private randomness?

   b) Lower bounds in $KT_0$ and $KT_1$ for IDs =\{1,2,...,n\}
2. Synch vs asynch: Is there really a separation?

3. Can optimal time $\tilde{O}(D+\sqrt{n})$ and $\tilde{O}(n)$ bits be achieved? Or lower bounds on time-bit trade-offs proved?

4. Building a shortest path tree in $o(m)$ bits?
Why it works

• Let $S$ be a subset of edges
• Imagine $2|S|$ equal sized intervals.

$t$ lands in some $I$, in top third or bottom third

Exactly one $x$ in $I$, in middle third
Why it works

• Let $S$ be a subset of edges
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Succeeds when parity of elements $< x$ is even and $t$ is $> x$
Or parity of elements $< x$ is odd and $t$ is $< x$
To summarize:

Spanning tree:
- Build in $O(n \log n)$ messages and time
- Use $\text{findany}$ to repair an ST in expected $O(n)$ messages, $O(\log n)$ local memory

Minimum spanning tree
- Build in $O(n \log n / \log \log n)$ messages and time
- Use $\text{findmin}$ to repair an MST in expected $O(n \log n / \log \log n)$ messages, $O(\log (n))$ local memory
Open problems and discussion

• Is $\Omega(m)$ communication required for building tree in asynchronous model?
• Prove separation of communication cost of findmin from findany
• Time v. communication tradeoffs
• Deterministic?
• Practical implementation: distances not exactly symmetric, topology, etc.
Findmin weight edge leaving

Two Tests to see if there is an edge leaving a comp:

1. Constant prob. **Test out**: 1 broadcast, 1 1-bit echo
2. High prob. **HP-Test out**: 1 broadcast, 1 $\log n$-bit echo
Findmin weight edge leaving

1. In parallel, with 1 Broadcast-and-echo do log n-wise search on weights with TestOut ‘s to find smallest wt interval

2. Verify its minimality with HP TestOut

3. Repeat if wrong or recurse on that interval
randomly choose a function

\( F: \text{edges} \rightarrow \{0,1\} \)

s.t. for a nonempty set \( S \),

w. constant prob. \( >0 \),

an ODD number of \( S \)'s elements map to 1.
Simple F (Thorup)

F has two parts,

- a 2-wise indep. hash function
  \[ h: U \rightarrow U \]
- \( t \), a random element of \( U \)

\[ F(s) = 1 \text{ iff } h(s) < t \]

F can be described in \( O(\log |U|) \) bits.
Why it works

• \( h \) hashes \( U \rightarrow U \), \( S \) subset of \( U \)
• Imagine \( 2|S| \) equal sized intervals.
• Repeat Testout in parallel $O(\log n)$ times to get prob error $1/n^c$?

• Would need to send $\log n$ hash functions

Or use deterministic amplification

Or…
Test 2: Set equality method to get HP-TestOut

A tree $T$ is maximal iff

the set $\text{Out}(T)$ of edges leaving all nodes in $T$

$= \text{the set } \text{In}(T) \text{ of edges entering all nodes in } T$
Test equality of two sets using polynomial ID-testing
$O(\log n)$ bits of communication

$f(x) = \prod_{a \in \text{Out}} (x-a)$
$g(x) = \prod_{b \in \text{In}} (x-b)$

Does $f(x) = g(x)$?

(Schwartz-Zippel):
Set $x = \text{random } \alpha \text{ in } \mathbb{Z}_p$, $p$ be a prime, $p > n^{c+2}$

compute over $\mathbb{Z}_p$:
$\Pr(f(\alpha) - g(\alpha)) = 0$ if $\text{In} \neq \text{Out}$

$= \Pr[\alpha = \text{root}]
= \#\text{roots} / p < n^2 / p = 1 / n^c$
Implementation (1 broadcast-and-echo)

- Leader broadcasts $\alpha$
- Each node $v$ computes $f_v(\alpha)$ and $g_v(\alpha)$ for its incident edges
- Starting at the leaves, pass to parent $f_v(\alpha) \cdot \pi_c f_c(\alpha)$, $c$ child of $v$
  $g$ is computed similarly
- Leader (root) computes $f-g$.
  - If 0, Testout is true, else false.
Open problem and discussion

• Is $\Omega(m)$ communication required for building MST in asynchronous model?
• Lower bound for findmin
• Time v. communication tradeoffs
• Applications to map reduce etc.

• Thank you
Algorithms for MST

Gallager, Humblet and Spira (JACM 1983)
• asynchronous
• works if starting with any number of nodes wake
• time $O(n^2)$, or $O(n \log n)$ if all nodes wake at start

$\tilde{O}(\sqrt{n+diameter})$ time in a synchronous model is possible with $\tilde{O}(m)$ communication. (Pandurangan, Robinson, Scquizzato, STOC 2017)
**Classic Result for MST:**
1983: $O(m + n \log n)$ bits

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Coordinator—private randomness

\[ k = \sqrt{n} \]

- Each node sends the name of each incident edge with probability \((\log n)/k\)

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