Random walks on dynamic graphs: Mixing times, hitting times, and return probabilities

Thomas Sauerwald and Luca Zanetti
8th Workshop on Advances in Distributed Graph Algorithms 2019
Outline

Intro

Random Walks on Sequences of Connected Graphs

Random Walks on Sequences of (Possibly) Disconnected Graphs

Conclusion
Random Walks on Graphs

A class of Markov chains where a particle is moving on the vertices of a graph:

- start from some specified vertex
- at each step, jump to a randomly chosen neighbor
Random Walks and Markov Chains

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Hitting Times (and Cover Times) on Static Graphs

Hitting and Cover Times

- Let $t_{hit}(u, v)$ be the expected time for a random walk to go from $u$ to $v$
- Let $t_{hit}(G) := \max_{u,v} t_{hit}(u, v)$ be the hitting time of the graph $G$
- Let $t_{cov}(G)$ the expected time to visit all vertices in $G$
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Some Classical Results:

- For any graph, $t_{hit}(G) \leq t_{cov}(G) \leq t_{hit} \cdot O(\log n)$ [Matthews, Annals of Prob.'88]
- For any graph, $t_{hit}(G) \leq t_{cov}(G) \leq 2 |E| (|V| - 1) = O(n^3)$ [Aleliunas, Karp, Lipton, Lovász and Rackoff, FOCS'79]
- For any graph, $t_{hit}(G) \leq t_{cov}(G) \leq 16 |E||V| \delta \Rightarrow t_{hit}(G) = O(n^2)$ if $G$ regular [Kahn, Linial, Nisan and Saks, J. Theoretical Prob.'88]
- For any graph, $t_{hit}(G) \leq (\frac{4}{27} + o(1)) \cdot n^3$ [Brightwell and Winkler, RSA'90]
- For any graph, $t_{cov}(G) \leq (\frac{4}{27} + o(1)) \cdot n^3$ [Feige, RSA'95]
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Motivation: Dynamic Graphs

Many prevalent networks are dynamically changing.
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Evolving, temporal or time-varying graph (Michail, Spirakis CACM'18; Kuhn, Oshman SIGACT News’11)
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Wireless/Mobile Networks
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Social Networks
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Wireless/Mobile Networks

(Distributed) Algorithms

Social Networks
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- Wireless/Mobile Networks
- (Distributed) Algorithms
- Social Networks
- Particle Processes

Big data scenario: Genome sequences for many species are available: each megabytes to gigabytes in size.

There are about 1 billion monthly active users in Facebook.
There are 5 billion global mobile phone users.

100 hours of videos uploaded per minute.
Random Walk on a Dynamic Graph Sequence

Lazy Random Walks

The random walk stays with probability 1/2 at the current location.
Random Walk on a Dynamic Graph Sequence

Lazy Random Walks

The random walk stays with probability $\frac{1}{2}$ at the current location.

$t = 1$
Random Walk on a Dynamic Graph Sequence

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The random walk stays with probability $1/2$ at the current location.

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Conclusion
We are interested in studying the following quantities on a sequence of dynamic graphs $\mathcal{G} = (G^1, G^2, \ldots)$ on a fixed set vertices:

- **Mixing time**: Number of steps needed for the distribution of the walk to become $\epsilon$-close to the stationary distribution.
- **Hitting times**: Expected number of steps to go from $u$ to $v$, $t_{\text{hit}}(u, v)$.

For static connected graphs:

- **Regular case**: $O(n^2)$ mixing and hitting times.
- **General case**: $O(n^3)$ mixing and hitting times.

For dynamic connected graphs:

- If $\pi(t)$ changes over time, in general, we don't have mixing.
- Can we at least say something about hitting times?
Agenda of this Talk

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Related Work: A Dichotomy for dynamic graphs

Avin, Koucky, and Lotker (ICALP’08, RSA’18)

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1. If $\pi(t)$ changes over time,
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   - mixing in $O(n^2 \log(n))$ steps
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Hitting Times can be bad! (The Sisyphus Graph)
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$$t = 1$$
Hitting Times can be bad! (The Sisyphus Graph)

\[
\begin{align*}
\text{\(n\)} - 2 & = \text{\(n\)} - 1 \\
\text{\(n\)} - 1 & = \text{\(n\)} - 2 \\
\text{\(n\)} - 3 & = \text{\(n\)} - 2
\end{align*}
\]
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\[ \begin{align*}
\text{t = 1} & : n - 1 \\
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$\begin{align*}
    n - 2 &\quad n - 3 \\
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    n &\quad 1 \\
    2 &\quad 3 \\
    4 &\quad n
\end{align*}$

$t = 1$

$\begin{align*}
    n - 3 &\quad n - 2 \\
    n - 2 &\quad n - 3 \\
    n &\quad 1 \\
    2 &\quad 3 \\
    4 &\quad n
\end{align*}$

$t = 2$

$\begin{align*}
    n - 4 &\quad n - 3 \\
    n - 3 &\quad n - 4 \\
    n &\quad 1 \\
    2 &\quad 3 \\
    4 &\quad n
\end{align*}$

$t = 3$
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\[
\begin{align*}
\text{\( t = 1 \)} & : & 1 & \rightarrow n & \rightarrow n-1 & & \rightarrow n-2 & \rightarrow n-3 & \rightarrow 4 & \rightarrow 3 & \rightarrow 2 & \rightarrow 1 \\
\text{\( t = 2 \)} & : & 1 & \rightarrow n & \rightarrow n-1 & & \rightarrow n-2 & \rightarrow n-3 & \rightarrow 3 & \rightarrow 2 & \rightarrow 1 \\
\text{\( t = 3 \)} & : & n & \rightarrow n-1 & \rightarrow n-2 & \rightarrow n & \rightarrow n-3 & \rightarrow n-4 & \rightarrow 1 & \rightarrow 2 & \rightarrow 3 \\
\text{\( t = 4 \)} & : & n & \rightarrow n-1 & \rightarrow n-2 & \rightarrow n & \rightarrow n-3 & \rightarrow n-4 & \rightarrow 1 & \rightarrow 2 & \rightarrow 3 
\end{align*}
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Random Walks on Sequences of Connected Graphs
Our Results

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Our Results

1. If all graphs are connected and regular,
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How can we derive these results?
Classical Proof (Spanning Tree Approach)

For any static graph $G$, $t_{cov}(G) \leq 2(n - 1)|E|$.

Aleliunas, Karp, Lipton, Lovász and Rackoff, FOCS’79
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Proof:
- Take a spanning tree $T$ in $G$
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- Take a spanning tree $T$ in $G$
- Consider a traversal that goes through every edge in $T$ twice
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**Proof:**
- Take a spanning tree $T$ in $G$
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- For any adjacent vertices $i, j$, $t_{hit}(i, j) + t_{hit}(j, i) \leq 2|E|$
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- For any adjacent vertices $i, j$, $t_{hit}(i, j) + t_{hit}(j, i) \leq 2|E|$ 
- Thus,

$$t_{cov}(G) \leq \sum_{(i, j) \in E(T)} t_{hit}(i, j) + t_{hit}(j, i)$$
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$$\leq 2(n - 1) \cdot |E|.$$
Classical Proof (Refinement based on Shortest Path)

(cf. Aldous, Fill’02)

For any static graph with diameter $D$, $t_{hit}(G) \leq 2|E| \cdot D$. 

Both proofs crucially rely on a static spanning tree or static shortest path!
For any static graph with diameter \( D \), \( t_{hit}(G) \leq 2|E| \cdot D \).

Proof:
- Fix two vertices \( s, t \), and consider a shortest path \( P = (u_0 = s, u_1, \ldots, u_l = t) \).
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For any static graph with diameter $D$, $t_{hit}(G) \leq 2|E| \cdot D$.

Proof:
- Fix two vertices $s$, $t$, and consider a shortest path $P = (u_0 = s, u_1, \ldots, u_l = t)$
- As before $t_{hit}(u_i, u_{i+1}) \leq 2|E|$. 
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This proves not only a bound of $O(n^3)$ for any graph, but also $O(n^2)$ for regular graphs.
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This proves not only a bound of $O(n^3)$ for any graph, but also $O(n^2)$ for regular graphs.

Both proofs crucially rely on a static spanning tree or static shortest path!
Return Times on Dynamic Graphs

A fundamental fact of the return times is that:

\[ t_{\text{hit}}(u, u) = \frac{1}{\pi(u)} = \frac{2|E|}{\deg(u)} \]

Is this true for dynamic graphs?
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Many combinatorial and probabilistic arguments seem to fail, but what about the \( t \)-step probabilities (and return probabilities)?
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm.

More precisely, 

$$\|p_t u, . - 1/n\|^2_2 \sim 1/\sqrt{t}$$

This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm. More precisely, 

$$\|p_t\|_2 \sim \frac{1}{\sqrt{t}}$$

This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm. More precisely,

$$\|p_t u_1 - \frac{1}{n}\|_2^2 \sim \frac{1}{\sqrt{t}}.$$ 

This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm is observed. More precisely, 

$$\|p_t\|_2 \sim \frac{1}{\sqrt{t}}$$

This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm.

More precisely, $\|p_t - \frac{1}{n}\|_2^2 \sim \frac{1}{\sqrt{t}}$

This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm.

More precisely, $\|p_t u, \cdot\|_2 \sim 1/\sqrt{t}$.

This property only requires each graph $G_t$ to be connected (and regular) at each step.

Diffusion of a Random Walk on a Static Cycle
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm is observed. More precisely, $\|p_t u\|_2 \sim 1/\sqrt{t}$. This property only requires each graph $G_t$ to be connected (and regular) at each step.

Random Walks on Sequences of Connected Graphs
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm. More precisely, $\|p_t u, - u/n\|_2^2 \sim 1/\sqrt{t}$.

This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm is achieved. More precisely, \[
\|p_t - \frac{1}{n}\|_2 \sim \frac{1}{\sqrt{t}}.
\] This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm is seen. More precisely, 

$$\|p_{t,u} - 1/n\|_2^2 \sim \frac{1}{\sqrt{t}}$$

This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm.

More precisely, $\|p_{t,u} - 1/n\|_2^2 \sim 1/\sqrt{t}$.

This property only requires each graph $G_t$ to be connected (and regular) at each step.

Step: 10
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm can be observed. More precisely,\

$$\|p_t\|_2 \sim \frac{1}{\sqrt{t}}$$

This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm is observed. More precisely, $\|p_t \|_2 \sim 1/\sqrt{t}$. This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm is observed. More precisely, $\|p_{t,u}\|_2 \sim 1/\sqrt{t}$ under these conditions. This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell^2$-norm is made. More precisely, 

$$\|p_t - u\|^2 \sim 1/\sqrt{t}$$

This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm $\|p(t) - \frac{1}{n}\|_2^2 \sim \frac{1}{\sqrt{t}}$. This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm.

More precisely, $\|p_t u, \cdot \|_2 \sim \frac{1}{\sqrt{t}}$.

This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm. More precisely, $\|p_t u, \cdot - 1/n\|_2 \sim 1/\sqrt{t}$. This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm can be observed. More precisely, 

$$\|p^n - \frac{1}{n}\|_2 \sim \frac{1}{\sqrt{t}}$$

This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm.

More precisely, $\|p_{t, u} - \frac{1}{n}\|^2_2 \sim \frac{1}{\sqrt{t}}$.

This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm. More precisely, $\|p_t - 1/n\|_2^2 \sim 1/\sqrt{t}$.

This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm. More precisely,

$$\| p_t u, - 1/n \|_2^2 \sim 1/\sqrt{t}$$

This property only requires each graph $G_t$ to be connected (regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm is made. More precisely, this property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm can be observed. More precisely,

$$\|p_{t,\cdot} - \frac{1}{n}\|_2^2 \sim \frac{1}{\sqrt{t}}$$

This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm.

More precisely, $\|p_{t-u}\|_2^2 \approx \frac{1}{\sqrt{t}}$.

This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm is expected. More precisely, 

$$\|p_{t} - \frac{1}{n}\|_2 \sim \frac{1}{\sqrt{t}}$$

This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm. More precisely, $\|p_t - \frac{1}{n}\|^2 \sim \frac{1}{\sqrt{t}}$

This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm can be achieved. More precisely, for each graph $G_t$ at each step $t$, the following holds:

$$\|p_{t,u} - \frac{1}{n}\|_2^2 \sim \frac{1}{\sqrt{t}}$$

This property only requires each graph $G_t$ to be connected (and regular) at each step $t$. For a random walk on a static cycle, the diagram illustrates the diffusion process at step 27.
Diffusion of a Random Walk on a Static Cycle

As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm. More precisely, $
abla p_t \leq 1/\sqrt{t}$.

This property only requires each graph $G_t$ to be connected (and regular) at each step.

Random Walks on Sequences of Connected Graphs
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm. More precisely, $\|p_t - \frac{1}{n}\|^2 \sim \frac{1}{\sqrt{t}}$. This property only requires each graph $G_t$ to be connected (and regular) at each step.

Step: 29
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm.

More precisely, $\| p_t \|_2 \sim 1/\sqrt{t}$.

This property only requires each graph $G_t$ to be connected and regular at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm.

More precisely, $\|p_{t,u} - 1/n\|_2^2 \sim \frac{1}{\sqrt{t}}$.

This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm can be expected. More precisely, $\|p_{t,u} - \frac{1}{n}\|^2 \sim \frac{1}{\sqrt{t}}$. This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm. More precisely, 

$$\|p_{t,v} - \frac{1}{n}\|_2^2 \sim \frac{1}{\sqrt{t}}$$

This property only requires each graph $G_t$ to be connected (regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm $\|p_t u \|_2 \approx \frac{1}{\sqrt{t}}$.

This property only requires each graph $G_t$ to be connected (and regular) at each step.
Diffusion of a Random Walk on a Static Cycle

As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm is expected. More precisely,

$$\|p_t u, - 1/n\|_2^2 \sim 1/\sqrt{t}$$

This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm is achieved. More precisely, \[ \|p_{t,x} - \frac{1}{n}\|_2 \sim \frac{1}{\sqrt{t}} \]

This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm is achieved. More precisely, $\|p_t u_\cdot - 1/n\|_2^2 \sim 1/\sqrt{t}$.

This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell^2$-norm
More precisely, $\|p_{t \cdot u} - \frac{1}{n}\|_2^2 \sim \frac{1}{\sqrt{t}}$

This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell^2$-norm can be achieved. More precisely,$$
abla \| p_t u, \cdot \|_2^2 \sim \frac{1}{\sqrt{t}}$$
This property only requires each graph $G_t$ to be connected and regular at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm.

More precisely, $\|p_t u, \cdot - 1/n\|_2 \sim 1/\sqrt{t}$.

This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm.

More precisely, $\|p^n(x) - \frac{1}{n}\|_2 \approx \frac{1}{\sqrt{t}}$

This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm is achieved.

More precisely, $\|p_{t, u} - \frac{1}{n}\|_2 \sim \frac{1}{\sqrt{t}}$.

This property only requires each graph $G_t$ to be connected (and regular) at each step.
Diffusion of a Random Walk on a Static Cycle

As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm is observed.

More precisely, $\|p_t \|_2 \sim 1/\sqrt{t}$

This property only requires each graph $G_t$ to be connected (and regular) at each step.
Diffusion of a Random Walk on a Static Cycle

As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm

More precisely,

$$\|p_{t^u} - 1/n\|_2 \sim 1/\sqrt{t}$$

This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm.

More precisely, $\|p_t u \|_2 \sim 1/\sqrt{t}$.

This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm.

More precisely, $\|p_{t,u} - \frac{1}{n}\|_2 \sim \frac{1}{\sqrt{t}}$.

This property only requires each graph $G_t$ to be connected (and regular) at each step.
Diffusion of a Random Walk on a Static Cycle

As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm is possible. More precisely, $\|p_t u_\cdot \cdot \cdot 1_n\|_2 \sim 1/\sqrt{t}$.

This property only requires each graph $G_t$ to be connected (and regular) at each step.

Random Walks on Sequences of Connected Graphs
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm.

More precisely, $\|p_t - \frac{1}{n}\|^2_2 \sim \frac{1}{\sqrt{t}}$

This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm is observed. More precisely, 

$$\|p_t u, \cdot - 1/n\|_2^2 \sim 1/\sqrt{t}$$

This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm is observed.

More precisely, $\|p_{t,u} - 1/n\|_2 \sim 1/\sqrt{t}$

This property only requires each graph $G_t$ to be connected (and regular) at each step.
As long as the probability mass is concentrated on a small set of vertices, substantial progress in the $\ell_2$-norm

More precisely, $\|p_{u,v}^t - \frac{1}{n}\|_2^2 \sim \frac{1}{\sqrt{t}}$

This property only requires each graph $G^t$ to be connected (and regular) at each step
Mixing in Dynamic Graphs: Definition

Sequence of (regular) graphs $\mathcal{G} = \{G^{(t)}\}_{t=1}^{\infty}$ on $V$ with transition matrices $\{P^{(t)}\}_{t=1}^{\infty}$

- $\pi P^{(t)} = \pi = 1/n$ for any $t$
Mixing in Dynamic Graphs: Definition

Sequence of (regular) graphs $\mathcal{G} = \{G^{(t)}\}_{t=1}^{\infty}$ on $V$ with transition matrices $\{P^{(t)}\}_{t=1}^{\infty}$

- $\pi P^{(t)} = \pi = 1/n$ for any $t$

$\ell_2$-mixing time

$$t_{\text{mix}}(\mathcal{G}) = \min \left\{ t \mid \sum_{y \in V} \left( P_{x,y}^{[0,t]} - \frac{1}{n} \right)^2 \leq \frac{1}{10n} \quad \forall x \in V \right\}. $$
Mixing in Dynamic Graphs: Definition

Sequence of (regular) graphs $G = \{G^{(t)}\}_{t=1}^\infty$ on $V$ with transition matrices $\{P^{(t)}\}_{t=1}^\infty$

- $\pi P^{(t)} = \pi = 1/n$ for any $t$

$\ell_2$-mixing time

$$t_{mix}(G) = \min \left\{ t \left| \sum_{y \in V} \left( P_{x,y}^{[0,t]} - \frac{1}{n} \right)^2 \leq \frac{1}{10n} \quad \forall x \in V \right. \right\}.$$ 

can be extended to non-regular graphs
Key Lemma

Let $P$ be the transition matrix of a random walk on a connected, regular graph $G = (V, E)$. Then for any probability distribution $\sigma$,

$$\sum_{u,v \in V} (\sigma(u) - \sigma(v))^2 \cdot P_{u,v} \gtrsim \left( \sum_{u \in V} \left( \sigma(u) - \frac{1}{n} \right)^2 \right)^2.$$
A Bound on the $\ell_2$-Decrease

Let $P$ be the transition matrix of a random walk on a connected, regular graph $G = (V, E)$. Then for any probability distribution $\sigma$,

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Proof Sketch:
As long as $\|\sigma - \frac{1}{n}\|^2_2$ is large $\Rightarrow$ $\sigma$ is concentrated on a small set of vertices.
A Bound on the $\ell_2$-Decrease

Key Lemma

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Proof Sketch:

As long as $||\sigma - \frac{1}{n}||_2^2$ is large $\Rightarrow \sigma$ is concentrated on a small set of vertices

$\Rightarrow \exists$ short path between $x^* = \arg\max_x \sigma(x)$ and $y$ s.t. $\sigma(y) \ll \sigma(x^*)$
A Bound on the $\ell_2$-Decrease

Let $P$ be the transition matrix of a random walk on a connected, regular graph $G = (V, E)$. Then for any probability distribution $\sigma$, 

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\sum_{u, v \in V} (\sigma(u) - \sigma(v))^2 \cdot P_{u, v} \gtrsim \left( \sum_{u \in V} \left( \sigma(u) - \frac{1}{n} \right)^2 \right)^2.
$$

**Key Lemma**

**Proof Sketch:**

As long as $\|\sigma - \frac{1}{n}\|_2^2$ is large $\Rightarrow$ $\sigma$ is concentrated on a small set of vertices

$\Rightarrow \exists$ short path between $x^* = \arg\max_x \sigma(x)$ and $y$ s.t. $\sigma(y) \ll \sigma(x^*)$

$\Rightarrow$ Let $\ell$ be the length of such path. Then,

$$
\sum_{u, v \in V} (\sigma(u) - \sigma(v))^2 P_{u, v} \geq \frac{(\sigma(x^*) - \sigma(y))^2}{2\ell} \text{ is large}
$$
Main Result (covering also non-regular graphs)

Let $G$ be a sequence of connected graphs of $n$ vertices with unique stationary distribution $\pi$. Moreover, denote with $\pi^* = \min_x \pi(x)$. Then:

$$t_{\text{mix}}(G) = O\left(\frac{n}{\pi^*}\right)$$

$$t_{\text{hit}}(G) = O\left(\frac{n \log n}{\pi^*}\right).$$

If all graphs in $G$ are regular, $t_{\text{hit}}(G) = O\left(\frac{n^2}{\pi^*}\right)$.

Theorem

To prove the bound on mixing:

**Key Lemma**

⇒ if $\ell_2$-norm is $\varepsilon$, after $O\left(\frac{n}{\pi^*\varepsilon}\right)$ steps it is less than $\varepsilon/2$.

⇒ Hence after $O\left(\frac{n}{\pi^*}\right)$ steps, $\ell_2$-norm will be small constant.

To prove the bound on hitting:

first obtain a refined bound on the $\ell_2$-norm decrease at each step

relate $t$-step probabilities to the $\ell_2$-norm in variance of the walk

use probabilistic arguments to relate $t$-step probabilities to hitting times.
Main Result (covering also non-regular graphs)

**Theorem**

Let $\mathcal{G}$ be a sequence of connected graphs of $n$ vertices with unique stationary distribution $\pi$. Moreover, denote with $\pi_* = \min_x \pi(x)$. Then:

- $t_{mix}(\mathcal{G}) = O(n/\pi_*)$
- $t_{hit}(\mathcal{G}) = O(n \log n / \pi_*)$.
- If all graphs in $\mathcal{G}$ are regular, $t_{hit}(\mathcal{G}) = O(n^2)$. 
Main Result (covering also non-regular graphs)

Let $\mathcal{G}$ be a sequence of connected graphs of $n$ vertices with unique stationary distribution $\pi$. Moreover, denote with $\pi_* = \min_x \pi(x)$. Then:

- $t_{\text{mix}}(\mathcal{G}) = O(n/\pi_*)$
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- If all graphs in $\mathcal{G}$ are regular, $t_{\text{hit}}(\mathcal{G}) = O(n^2)$.

To prove the bound on mixing:
Main Result (covering also non-regular graphs)

Let $\mathcal{G}$ be a sequence of connected graphs of $n$ vertices with unique stationary distribution $\pi$. Moreover, denote with $\pi_* = \min_x \pi(x)$. Then:

- $t_{\text{mix}}(\mathcal{G}) = O(n/\pi_*)$
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- If all graphs in $\mathcal{G}$ are regular, $t_{\text{hit}}(\mathcal{G}) = O(n^2)$.

To prove the bound on mixing:

- Key Lemma $\Rightarrow$ if $\ell_2$-norm is $\varepsilon$, after $O(n/(\pi_*\varepsilon))$ steps it is less than $\varepsilon/2$
- Hence after $O(n/\pi_*)$ steps, $\ell_2$-norm will be small constant $\Rightarrow$ walk mixed
Main Result (covering also non-regular graphs)

**Theorem**

Let $G$ be a sequence of connected graphs of $n$ vertices with unique stationary distribution $\pi$. Moreover, denote with $\pi_* = \min_x \pi(x)$. Then:

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To prove the bound on hitting:
Main Result (covering also non-regular graphs)

Let $G$ be a sequence of connected graphs of $n$ vertices with unique stationary distribution $\pi$. Moreover, denote with $\pi_* = \min_x \pi(x)$. Then:

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To prove the bound on mixing:

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To prove the bound on hitting:

- first obtain a refined bound on the $\ell_2$-norm decrease at each step
- relate $t$-step probabilities to the $\ell_2$-norm in variance of the walk
- use probabilistic arguments to relate $t$-step probabilities to hitting times
What happens when the connectivity properties of the graph change over time?
How to bound mixing when connectivity is intermittent

- In static graphs, the eigenvalues of the individual transition matrices give a good bound on mixing:

\[ \frac{1}{1 - \lambda} \lesssim t_{\text{mix}}(G) \lesssim \frac{\log(n)}{1 - \lambda} \]
How to bound mixing when connectivity is intermittent

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- This is not necessarily true for dynamic graphs:
How to bound mixing when connectivity is intermittent

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- This is not necessarily true for dynamic graphs:
  \[
  \begin{align*}
    \text{Odd } t & \quad 1 - \lambda(P^{(t)}) = 0 \\
    \text{Even } t & \quad 1 - \lambda(P^{(t)}) = 0
  \end{align*}
  \]
How to bound mixing when connectivity is intermittent

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Even \( t \)

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Average transition probabilities

Odd $t$: $1 - \lambda(P^{(t)}) = 0$

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Average transition probabilities

Odd $t$: $1 - \lambda(P^{(t)}) = 0$

Even $t$: $1 - \lambda(P^{(t)}) = 0$

$1 - \lambda(\overline{P}) = \Omega(1)$
Mixing based on average connectivity properties

Consider a sequence \( \mathcal{G} \) with transition matrices \( \{P^{(t)}\}_{t=1}^{\infty} \) such that

1. \( \pi P^{(t)} = \pi \) for any \( t \)

2. there exists a time window \( T \geq 1 \) such that, for any \( i \geq 0 \), \( \overline{P}^{[i \cdot T + 1, (i+1) \cdot T]} \)
   is ergodic with spectral gap greater or equal than \( 1 - \lambda \)

Then, \( t_{\text{mix}}(\mathcal{G}) = O(T^2 \log(1/\pi_{\star})/(1 - \lambda)) \)
Mixing based on average connectivity properties

**Theorem**

Consider a sequence $\mathcal{G}$ with transition matrices $\{P^{(t)}\}_{t=1}^{\infty}$ such that

1. $\pi P^{(t)} = \pi$ for any $t$

2. there exists a time window $T \geq 1$ such that, for any $i \geq 0$, $P^{[i \cdot T + 1, (i+1) \cdot T]}$ is ergodic with spectral gap greater or equal than $1 - \lambda$

Then, $t_{\text{mix}}(\mathcal{G}) = O(T^2 \log(1/\pi_*)/(1 - \lambda))$

**Corollary**

Suppose that for any time window $\mathcal{I} = [i \cdot T + 1, (i+1) \cdot T]$ and any subset of vertices $A \subseteq V$ there exists $i \in \mathcal{I}$ such that $\Phi_{P^{(i)}}(A) \geq \phi$. Then,

$$t_{\text{mix}}(\mathcal{G}) = O(T^3 \log(1/\pi_*)/\phi^2)$$
Mixing based on average connectivity properties

**Theorem**

Consider a sequence $\mathcal{G}$ with transition matrices $\{P^{(t)}\}_{t=1}^{\infty}$ such that

1. $\pi P^{(t)} = \pi$ for any $t$
2. there exists a time window $T \geq 1$ such that, for any $i \geq 0$, $\overline{P}^{[i \cdot T + 1, (i+1) \cdot T]}$ is ergodic with spectral gap greater or equal than $1 - \lambda$

Then, $t_{\text{mix}}(\mathcal{G}) = O(T^2 \log(1/\pi_*)/(1 - \lambda))$

**Corollary**

Suppose that for any time window $\mathcal{I} = [i \cdot T + 1, (i + 1) \cdot T]$ and any subset of vertices $A \subseteq V$ there exists $i \in \mathcal{I}$ such that $\Phi_{P^{(i)}}(A) \geq \phi$. Then,

$$t_{\text{mix}}(\mathcal{G}) = O(T^3 \log(1/\pi_*)/\phi^2)$$

Since $t_{\text{hit}}(\mathcal{G}) = O(t_{\text{mix}}(\mathcal{G})/\pi_*)$, does polynomial mixing time imply polynomial hitting times?
Mixing based on average connectivity properties

**Theorem**

Consider a sequence $G$ with transition matrices $\{P(t)\}_{t=1}^{\infty}$ such that

1. $\pi P(t) = \pi$ for any $t$
2. there exists a time window $T \geq 1$ such that, for any $i \geq 0$, $P^{[i \cdot T+1, (i+1) \cdot T]}$ is ergodic with spectral gap greater or equal than $1 - \lambda$

Then, $t_{mix}(G) = O(T^2 \log(1/\pi_*)/(1 - \lambda))$

**Corollary**

Suppose that for any time window $I = [i \cdot T + 1, (i+1) \cdot T]$ and any subset of vertices $A \subseteq V$ there exists $i \in I$ such that $\Phi_{P(i)}(A) \geq \phi$. Then,

$$t_{mix}(G) = O(T^3 \log(1/\pi_*)/\phi^2)$$

Since $t_{hit}(G) = O(t_{mix}(G)/\pi_*)$, does polynomial mixing time imply polynomial hitting times?

- NO! When the graphs are disconnected, $\pi_*$ can be exponentially small
Mixing based on average connectivity properties

**Theorem**

Consider a sequence $G$ with transition matrices $\{P^{(t)}\}_{t=1}^{\infty}$ such that
1. $\pi P^{(t)} = \pi$ for any $t$
2. there exists a time window $T \geq 1$ such that, for any $i \geq 0$, $P^{[i \cdot T + 1, (i+1) \cdot T]}$ is ergodic with spectral gap greater or equal than $1 - \lambda$

Then, $t_{mix}(G) = O(T^2 \log(1/\pi_*)/(1 - \lambda))$

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Suppose that for any time window $I = [i \cdot T + 1, (i+1) \cdot T]$ and any subset of vertices $A \subseteq V$ there exists $i \in I$ such that $\Phi_{P^{(i)}}(A) \geq \phi$. Then,

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Since $t_{hit}(G) = O(t_{mix}(G)/\pi_*)$, does polynomial mixing time imply polynomial hitting times?

- **NO!** When the graphs are disconnected, $\pi_*$ can be exponentially small
- Why? We can simulate a random walk on a directed graph:
Simulating a Directed Graph using Dynamic Graphs

Random Walk Behaviour:

Since the stationary distribution is exponentially small for the vertices at the bottom, hitting time is exponential in $n$. However, average transition matrix $P$ can be easily made ergodic (add same cycle of $n - 2$ matrices in reverse order) $\Rightarrow$ mixing time polynomial in $n$ by our theorem!
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$t = 1$

Random Walks on Sequences of (Possibly) Disconnected Graphs
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$t = 4$
Simulating a Directed Graph using Dynamic Graphs

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$t = 5$
Simulating a Directed Graph using Dynamic Graphs

Random Walk Behaviour:
Since the stationary distribution is exponentially small for the vertices at the bottom, hitting time is exponential in $n$. However, average transition matrix $P$ can be easily made ergodic (add same cycle of $n - 2$ matrices in reverse order) $\Rightarrow$ mixing time polynomial in $n$ by our theorem!

$t = 6$
Simulating a Directed Graph using Dynamic Graphs

Random Walk Behaviour:

Since the stationary distribution is exponentially small for the vertices at the bottom, hitting time is exponential in $n$. However, average transition matrix $P$ can be easily made ergodic (add same cycle of $n-2$ matrices in reverse order) $\Rightarrow$ mixing time polynomial in $n$ by our theorem!

$t = 7$
Simulating a Directed Graph using Dynamic Graphs

Random Walk Behaviour:
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$t = 8$
Simulating a Directed Graph using Dynamic Graphs

Random Walk Behaviour:

Since the stationary distribution is exponentially small for the vertices at the bottom, hitting time is exponential in \( n \). However, average transition matrix \( \mathbf{P} \) can be easily made ergodic (add same cycle of \( n - 2 \) matrices in reverse order) \( \Rightarrow \) mixing time polynomial in \( n \) by our theorem!
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Random Walk Behaviour:
Since the stationary distribution is exponentially small for the vertices at the bottom, hitting time is exponential in $n$. However, the average transition matrix $P$ can be easily made ergodic (add the same cycle of $n-2$ matrices in reverse order) $\Rightarrow$ mixing time polynomial in $n$ by our theorem!

$t = 1$
Simulating a Directed Graph using Dynamic Graphs

Random Walk Behaviour:
Since the stationary distribution is exponentially small for the vertices at the bottom, hitting time is exponential in $n$. However, average transition matrix $P$ can be easily made ergodic by adding a cycle of $n-2$ matrices in reverse order. $\Rightarrow$ mixing time polynomial in $n$ by our theorem!
Simulating a Directed Graph using Dynamic Graphs

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Since the stationary distribution is exponentially small for the vertices at the bottom, hitting time is exponential in $n$. However, average transition matrix $P$ can be easily made ergodic (add same cycle of $n-2$ matrices in reverse order) $\Rightarrow$ mixing time polynomial in $n$ by our theorem!

$t = 3$
Simulating a Directed Graph using Dynamic Graphs

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Simulating a Directed Graph using Dynamic Graphs

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- Since the stationary distribution is exponentially small for the vertices at the bottom, hitting time is \textit{exponential} in \( n \)

- However, average transition matrix \( \overline{P} \) can be easily made ergodic (add same cycle of \( n - 2 \) matrices in reverse order)

\( \Rightarrow \) mixing time \textit{polynomial} in \( n \) by our theorem!
Outline

Intro

Random Walks on Sequences of Connected Graphs

Random Walks on Sequences of (Possibly) Disconnected Graphs

Conclusion
Conclusions and Future Work

We have exhibited a dichotomy for random walks on dynamic graphs:

\[ G(n, p) \]

random changes: dynamic version of Random Graphs

bounded changes: edge set changes by a small number at each step

But: In real-world graphs, also the vertex set may change!
Conclusions and Future Work

We have exhibited a dichotomy for random walks on dynamic graphs:

- If stationary distribution does not change over time, behaviour is comparable to static graphs
- otherwise, they lose many nice properties associated with random walks on static graphs (even when the changes in the stationary distribution are small, e.g., all graphs are bounded-degree)
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THANK YOU