Pseudorandomness: Some Distributed Applications

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What this talk will be about

Goals of this talk

- Introduce several key ideas about pseudorandomness using a toy example.

  Existence via Probabilistic Method, connections to error correcting codes, hash functions.

- See (up to 3) examples how these ideas have been used in distributed computing.

  Low-congestion sampling, derandomization.

- See some extra pseudorandom objects on the way.

  Expander graphs, pairwise independent hash functions.

What is pseudorandomness?

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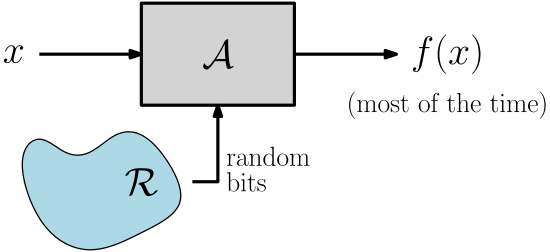

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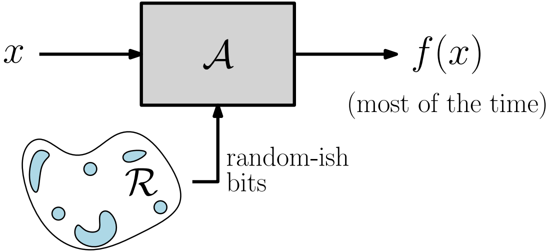

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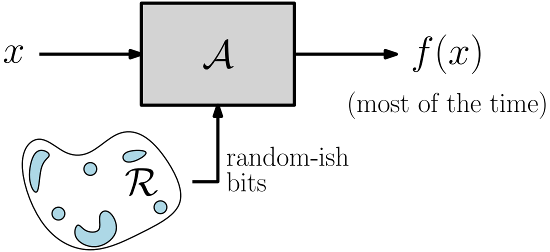

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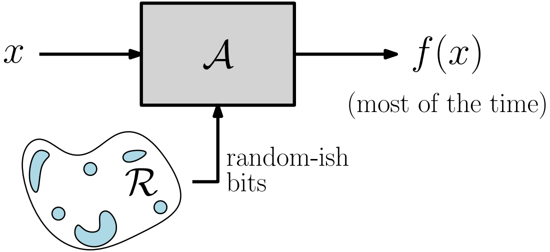

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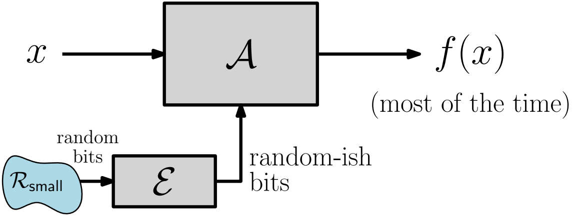

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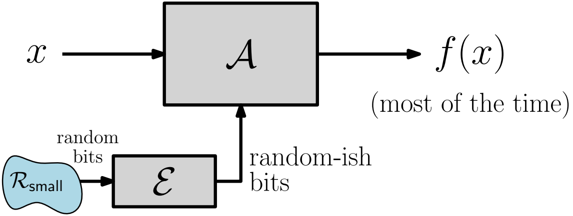

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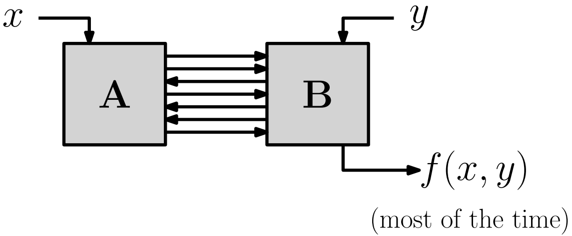
Furthermore: is the sparsification **reasonably computable**?

Pseudorandomness = study of when this is possible.

Example: Equality in 2-party Communication Complexity



Take $f = \mathsf{EQ}_n$: $\mathsf{EQ}_n(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$ with $x, y \in \{0, 1\}^n$.

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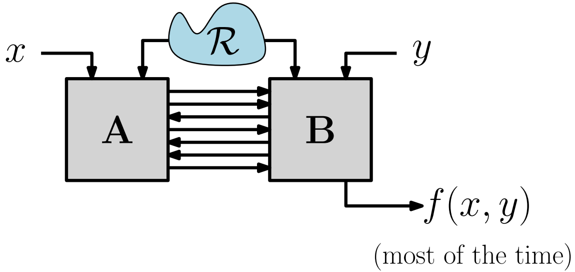

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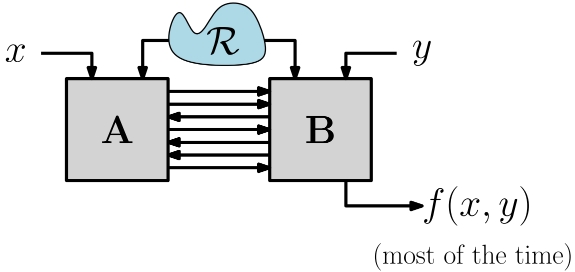

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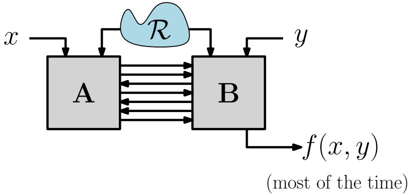
For error $\epsilon < 1/2$, simple algorithm using $\lceil \log(1/\epsilon) \rceil$ bits?

Algorithm for Equality with shared randomness



1. Repeat for t in $[\lceil \log(1/\epsilon) \rceil]$
   1.1 Alice reads $r^{(t)} \in \{0,1\}^n$ from shared randomness.
   1.2 Alice computes $\langle x, r^{(t)} \rangle = \sum_{i=1}^{n} x_i \cdot r_i^{(t)} \mod 2$, sends it to Bob.
2. Bob outputs "`Equal`" iff $\langle x, r^{(t)} \rangle = \langle y, r^{(t)} \rangle, \forall t \in [\lceil \log(1/\epsilon) \rceil]$

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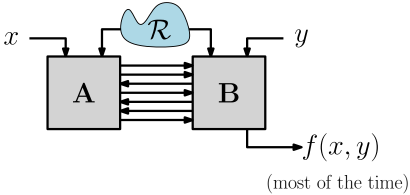

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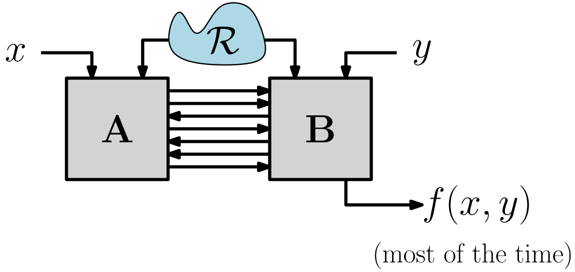
Proof: when $x \neq y$, each $r^{(t)}$ has a probability $1/2$ to be s.t. $\langle x, r^{(t)} \rangle \neq \langle y, r^{(t)} \rangle$ (to see it, focus on any index i s.t. $x_i \neq y_i$).

Do we need that many random strings?



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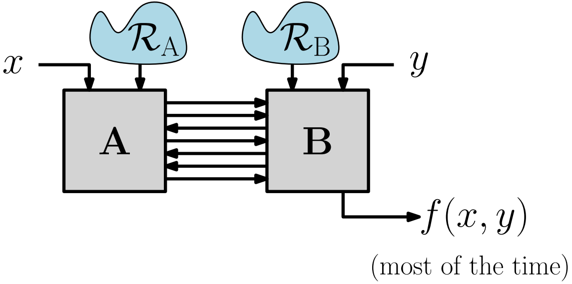

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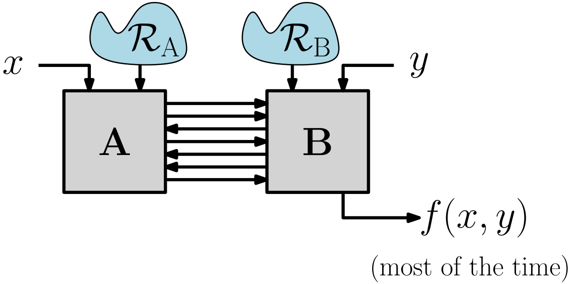

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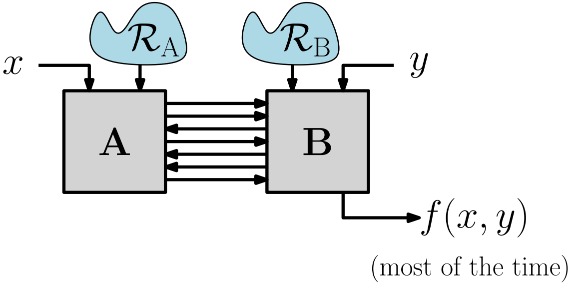

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Can we identify a subset of the randomness that works?

First Important Idea: probabilistic method

Goal: find subset $R \subseteq \{0,1\}^n$ such that:

- (small) $|R| \ll 2^n$,
- (useful) $\forall x, y \in \{0,1\}^n, x \neq y, \Pr_{r \in R}[\langle x, r \rangle \neq \langle y, r \rangle] \geq 1/4$.

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Proof: Pick specific pair x, y with $x \neq y$. In k random strings $r_1, \ldots, r_k$, $k/2$ in expectation are s.t. $\langle x, r\rangle \neq \langle y, r\rangle$.

Probability that less than $k/4$ strings are s.t. $\langle x, r\rangle \neq \langle y, r\rangle$ is bounded by: $\exp(-k/12)$ (Chernoff bound)

Probality that less than $k/4$ strings are s.t. $\langle x, r\rangle \neq \langle y, r\rangle$ for some pair x, y: $2^{2n} \cdot \exp(-k/12) < 1$ for $k > (24\ln(2)) \cdot n$.

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Probabilistic method at its fullest: pseudorandom generators

More generally: Consider any randomized algorithm with N inputs, failure probability ϵ. We can identify a subset of its randomness of size $O(\epsilon^{-1} \log N)$ such that the error is at most 2ϵ when using random strings from this sparse randomness.

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Pseudorandom generators: generators of such strings. An explicit construction would give $\mathsf{P} = \mathsf{BPP}$, by enumerating through all the strings.

Second Important Idea: connection to other objects

Pick a set of random strings $R \subseteq \{0,1\}^n$ like we just constructed, i.e., for any $x, y \in \{0,1\}^n$ such that $x \neq y$:

$$\{r \in R : \langle x, r \rangle \neq \langle y, r \rangle\} \geq |R|/4 \ .$$

For each $x \in \{0,1\}^n$, consider the $|R|$-bit word $w_R(x)$:

$$w_R(x) = \langle x, r_1 \rangle . \langle x, r_2 \rangle . \ldots . \langle x, r_{|R|} \rangle \ .$$

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With parameters $[24 \ln(2) n, n, 6 \ln(2) n]$, over $\mathbb{F}_2$.

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Conversely: Can get more randomness-efficient algorithms from results on Error Correcting Codes.

Second Important Idea continued: another connected object

We can also see each $r \in R$ as a map h_r: $\{0,1\}^n \to 0, 1$, with $h_r(x) = \langle x, r \rangle$

Definition ((ϵ-almost) Pairwise-Independent Hash Functions)

A set $\mathcal{H}$ of functions $h : [N] \to [M]$ s.t.:
$\forall x \neq y \in [N]^2, z, z' \in [M]^2$,
$|\Pr_{h \in \mathcal{H}}[h(x) = z \wedge h(y) = z'] - 1/M^2| \leq \epsilon/M^2$.

There exists a family of such hash functions of size $\text{poly}(M, 1/\epsilon)(\log N / \log\log N)$.

Lessons learned so far

Given ANY randomized algorithm, we can reduce its use of randomness to $O(\log n)$ bits without affecting its success/failure probability too much.

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However this is existential. While existence of some pseudorandom object can often be done by probabilistic method, finding an explicit construction can be harder.

Going the last mile requires finding some explicit structure that works in the given application (error correcting codes, families of hash functions, fields, polynomials, classes of graphs).

Distributed applications

Distributed application 1: random sampling

Setting: CONGEST model:

- Graph $G = (V, E)$ is the input and the communication network.
- $n =$ number of nodes, $\Delta =$ maximum degree. Known by all nodes.
- In a round, each node can send $O(\log n)$ bits on each incident edge.
- Complexity is the number of rounds to achieve the wanted result.

LOCAL model: same with infinite bandwidth.

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- k-coloring: each node must be properly colored with a value from $\{1, \ldots, k\}$.
- k-list-coloring: each node starts with a list of k colors, from which it must choose its color.
- palette Ψ_v: colors still available to a node v as it avoids colors already chosen by neighbors.
- edge-coloring: we have to color the edges instead of the nodse

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Highlights from [HN23, HNT22], lowering the complexity in CONGEST to that in LOCAL (or almost):

- In $O(\log^* n)$: $(1+\epsilon)\Delta$ coloring, $(1+\epsilon)\Delta$ edge-coloring, when $\Delta \in \Omega(\log^{1+1/\log^*} n)$. [HN23]
- $\deg +1$-list-coloring in $O(\log^5 \log n)$ [HNT22], $(2\Delta - 1)$-edge coloring in $O(\log^4 \log n)$ rounds. [HN23]

  (saving an additive $\log \Delta$ in both cases)

Fast coloring with an excess of colors [SW10]

- Consider the nodes v s.t. $|\Psi_v| \geq 2x \cdot d_v$.
  (i.e., with $\times 2x$ more available colors than active neighbors)
- Let each such node try x colors.
- They each get colored w.p. $1 - 2^{-x}$.

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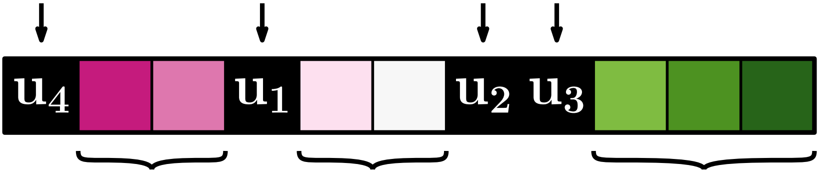

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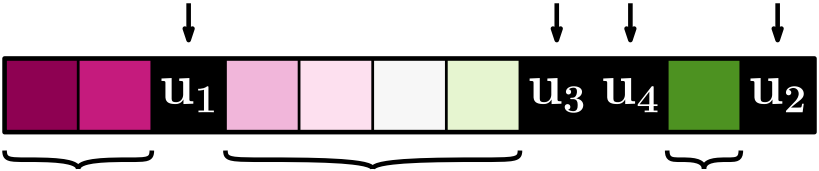

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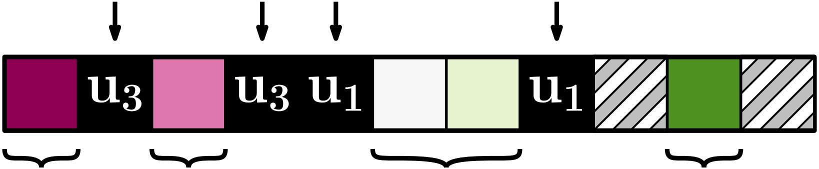

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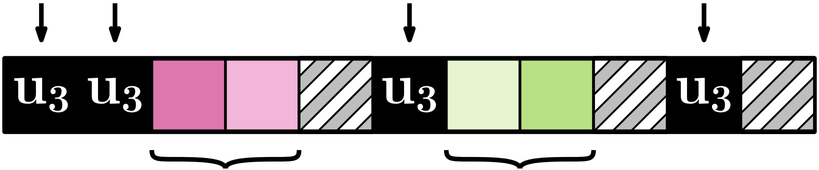

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Algorithm for 2Δ-coloring in LOCAL, for a node v, with neighbors of higher ID $N^+(v)$.

1. Set $x \leftarrow 1$.
2. For $i = 1$ to $O(\log^* n)$
   2.1 Repeat $O(1)$ times:
      2.1.1 v picks a set S_v of x random colors in its palette Ψ_v.
      2.1.2 v sends S_v to $N(v)$, computes $T_v = S_v \setminus \bigcup_{u \in N^+(v)} S_u$.
      2.1.3 If $T_v \neq \emptyset$, v permanently adopts a color in T_v.
      It sends its final color to its neighbors and stops the algorithm.
   2.2 Set $x \leftarrow \min(2^x, \log n)$.
3. If $\Delta < O(\log^{1+1/\log^* n} n)$, finish the coloring with a deterministic algorithm. [GK21]

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Proof idea: the degree of each node is bounded by $\max(\frac{2\Delta}{x}, O(\log n))$, w.h.p., throughout the algorithm.

Doing the same in CONGEST: the cost of sending a set

Basic argument of the algorithm is:

- given a space of colors $\mathcal{C}$, and
- a subset $T \subseteq \mathcal{C}$ of colors to avoid of size $|T| \leq |\mathcal{C}|/2$,
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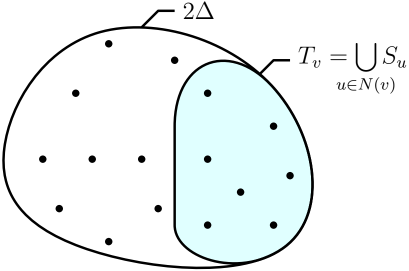

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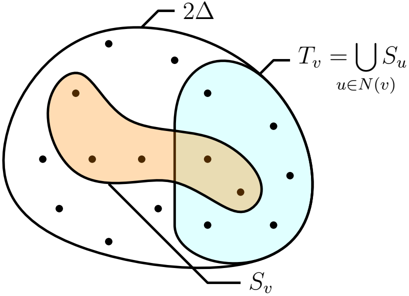

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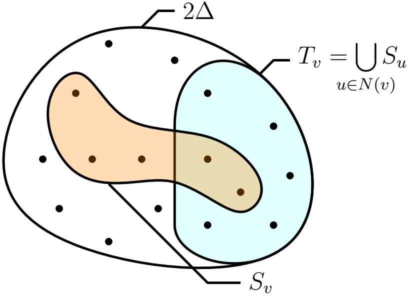

Issue for Congest: describing an arbitrary set $S \subseteq [2\Delta]$, $|S_v| \in [1, \Theta(\log n)]$ requires $\Theta(\log n \cdot \log \Delta)$ bits.

Solution 1 (existential), by probabilistic method

Same proof scheme as before:

- Fix $T \subseteq [2\Delta]$, $|T| \leq \Delta$.
- For a random set $S \subseteq [2\Delta]$ of size x, $\Pr[S \subseteq T] \leq 2^{-x}$.
- Taking k random sets $S_1 \dots S_k \subseteq [2\Delta]$, $\leq k \cdot 2^{-x}$ are expected to be $\subseteq T$.
- The probability that $\geq 2 \cdot k2^{-x}$ of the sets are $\subseteq T$ is $\leq e^{-k \cdot 2^{-x}/3}$.
- There are $\leq 2^{2\Delta}$ possible sets T.
- With $x \in O(\log n)$, $k \in \text{poly}(\log \Delta, n)$, k random sets are s.t. $T \subseteq [2\Delta]$, $|T| \leq \Delta$, $|\{i : S_i \subseteq T\}| \leq 2k2^{-x}$ with probability > 0.

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Can strengthen it a bit: For any sufficiently large set T, most sets $S_1 \dots S_{\text{poly}(\log \Delta, n)}$ intersect T in $\approx S_i \cdot |T|/(2\Delta)$ elements.

Representative sets

Lemma (Representative sets)

Let U be a universe of size k. A family $\mathcal{F} = \{S_1, \ldots, S_t\}$ of s-sized sets is said to be an (α, δ, ν)-representative family iff:

$$\forall T \subseteq U, |T| \geq \delta k : \quad \Pr_{i \in_u [t]}\left[\left|\frac{|S_i \cap T|}{|S_i|} - \frac{|T|}{k}\right| \leq \alpha \frac{|T|}{k}\right] \geq (1-\nu), \quad (1)$$

$$\forall T \subseteq U, |T| < \delta k : \quad \Pr_{i \in_u [t]}\left[\frac{|S_i \cap T|}{|S_i|} - \delta \leq \alpha\delta\right] \geq (1-\nu), \quad (2)$$

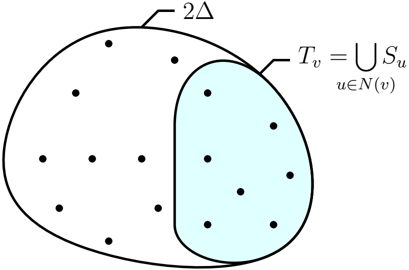
$$\forall u \in U : \quad \Pr_{i \in_u [t]}[u \in S_i] \in [1-\alpha, 1+\alpha]\frac{s \cdot t}{k}. \quad (3)$$

Such families exist for $t \in \Theta(k/\nu + k \log k)$ and $s \in \Theta(\alpha^{-2}\delta^{-1}\log(1/\nu))$.

Proof: probabilistic method.

Solution 2 (explicit), using expander graphs

We add a graph structure on the space of colors.



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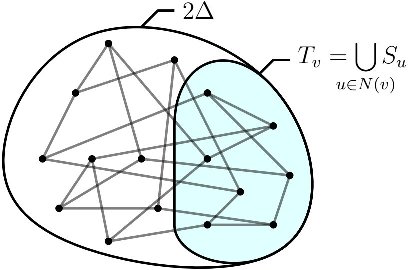

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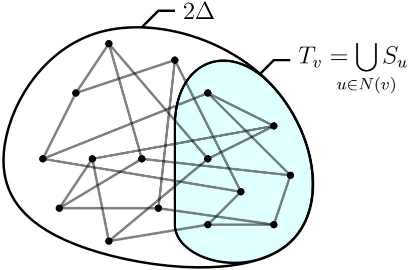

On a graph of degree $O(1)$, describing a walk of length k takes $O(\log \Delta)$ (to describe the starting vertex) $+O(k)$ bits (to describe the k steps taken from this starting node).

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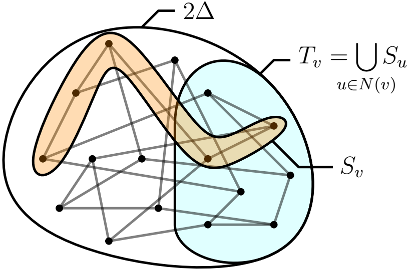

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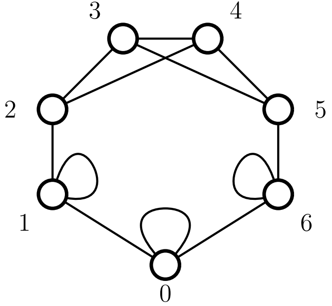
But why should such sets have the right properties?

Spectral expanders and random walks



1	1	0	0	0	0	1
1	1	1	0	0	0	0
0	1	0	1	1	0	0
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0	0	1	1	0	1	0
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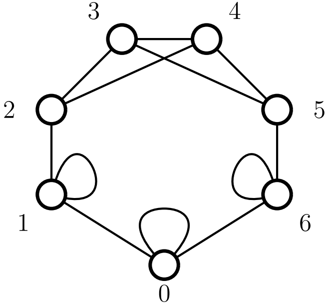
Consider a connected, non-bipartite, regular digraph G, transition matrix M_G.

Consider eigenvalues of M_G $\lambda_1, \ldots, \lambda_n$, s.t. $|\lambda_i| \geq |\lambda_{i+1}|$

The largest eigenvalue is always $\lambda_1 = 1$ and corresponds to the uniform distribution, others are < 1.

Spectral expanders and random walks



$$\begin{matrix} 1/3 & 1/3 & 0 & 0 & 0 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 1/3 & 1/3 & 0 \\ 0 & 0 & 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 0 & 0 & 0 & 1/3 & 0 \end{matrix}$$

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Expanders mix well

The spectral gap $\gamma = 1 - |\lambda_2|$ captures how fast and close a random walk gets to the uniform distribution.

Theorem ([Hea08, WX05])

Consider an expander graph G with spectral gap $\gamma = 1 - |\lambda_2|$, and a subset $S \subseteq V$ of its nodes. Consider a random walk of length k, and let X measure how many steps of the walk are in S. For any $\epsilon > 0$:

$$\Pr\left[\left|\frac{1}{k}X - \frac{|S|}{n}\right| \geq \epsilon\right] \leq 2e^{-\epsilon^2\gamma k/4}$$

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Just need to find an expander over 2Δ nodes, with constant degree, spectral gap $\Omega(1)$.

Existence of expanders

Do expanders like we want exist?

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But explicit constructions are also known.

Graphs defined by the structure of a prime fields, recursive constructions...

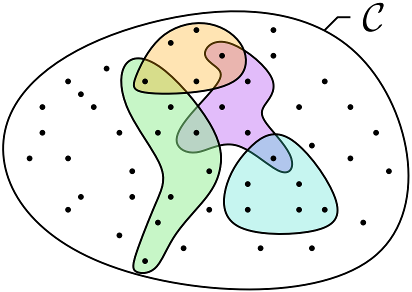
Consequences [HN23]

Δ	Coloring tasks	Complexity in CONGEST
$\Omega(\log^{1+1/\log^* n} n)$	$(1+\epsilon)\Delta$-vertex $(1+\epsilon)\Delta$-edge	$O(\log^* n)$
$O(\log^{1+1/\log^* n} n)$	$(1+\epsilon)\Delta$-vertex	$O(\log^3 \log n)$
	$(2\Delta - 1)$-edge	$O(\log^4 \log n)$
$\Omega(\sqrt{\log^{1+1/\log^* n} n})$	$(1+\epsilon)\Delta^2$-vertex distance-2	$O(\log^* n)$
$O(\sqrt{\log^{1+1/\log^* n} n})$		$O(\log^4 \log n)$
$\Omega(\log^{1+1/c'} n)$	$\Delta \log^{(c)}$-vertex $\Delta \log^{(c)}$-edge	$O(1)$
$\Omega(\sqrt{\log^{1+1/c'} n})$	$\Delta^2 \log^{(c)} n$-vertex distance-2	

Techniques: Rödl nibble [DGP98] (for $(1+\epsilon)\Delta$-edge coloring), shattering [BEPS16] and deterministic algorithm [GK21] (for smaller Δ).

Harder case: non-shared color space [HNT22]

Suppose now that each node has a **list** of 2Δ colors in a much bigger universe of colors $\mathcal{C}$.



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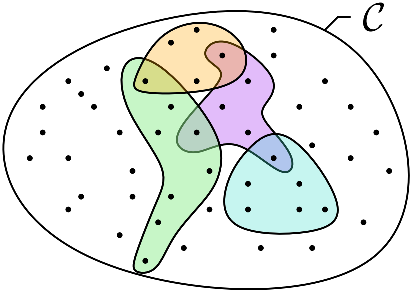

The expander construction no longer works here. How do we extend the argument?

Existential solution: pure hashing

Suppose nodes could sample and communicate a perfect hash function $\mathcal{C} \to [4\Delta]$. To try colors, nodes could:

1. Consider the part of their palette that hash to a value $\leq \Theta(\log n)$. ($\Theta(\log n)$ do, in expectation)
2. Send a $\Theta(\log n)$ bitmap to say "I'm trying colors with these hashes"

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Extends all previous results to the list-setting, but with very large local computation.

Doing without the existential arguments

1. Each node v finds an $\epsilon/2$-almost-pairwise independent hash function $h_v : \mathcal{C} \to [4\Delta/\epsilon]$ with $\leq \epsilon\Delta$ collisions.
2. Each node samples colors/hashes using the expander graph structure applied to the hash space $[2\Delta/\epsilon]$.
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General idea: each node has at least $(2 - \epsilon)\Delta$ colors which are not in collision with its hash function.

Each color tried by a neighbor removes at most one possible hash for v. As before, nodes sample more and more colors but always guaranteeing that each node has a constant fraction of its palette untried by neighbors in any given round.

Each node, when sampling hashes, should thus succeed in finding a color with the usual probability.

The point of it

Allowed us to get a $\text{poly} \log \log n$ algorithm for $\deg + 1$-list-coloring, and the list-coloring versions of all previously mentionned problems (with extra colors).

Also, the algorithm for $\deg + 1$-list-coloring finishes in $O(\log^* n)$ for $\Delta > \log^7 n$.

Also useful in subsequent work: see distance-2 paper at this DISC [FHN23].

Distributed application 2: derandomization

Pseudorandomness is also essential to derandomizing randomized algorithms: less random bits to deterministically choose, the better!

- MPC [CDP21a, CDP21b, CC22, FGG22, CCDM23a]
- Congested Clique [CPS20, CDP21c, CCDM23b]

Concluding

To go further if this has piqued your interest, “Pseudorandomness” by Salil Vadhan is a great survey. [Vad12]

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Some open questions:

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  The explicit solution presented before works for distance-1 node-coloring, not some other settings (edge-coloring, distance-2 coloring).

- For the MPC results relying on pseudorandom generators, do without them.

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Thanks!

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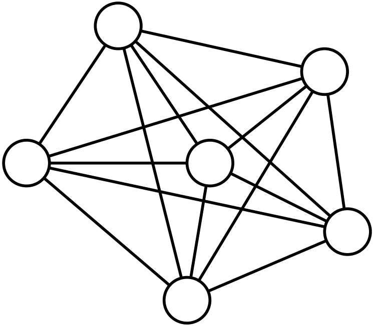
Creating slack: sparsity



Intuition: if your neighbors are mostly disconnected, it's likely some of them will pick the same color.

Formally: $\zeta_v = \frac{1}{\Delta}\left(\binom{\Delta}{2} - |E[N(v)]|\right)$.

Theorem ([EPS15]): Let each node try a random color, then afterwards:

$$\Pr[s_v \leq c \cdot \zeta_v] \leq \exp(-c' \cdot \zeta_v).$$

for some universal constants c and c'.

Nodes of the line-graph L_G ($e \in E_G$ iff $v_e \in V_{L_G}$; $v_e v_{e'} \in E_{L_G}$ iff $e = uv, e' = uv'$) have sparsity $\Omega(\Delta)$. **In an edge-coloring setting, [EPS15]'s result gives $\Omega(\Delta)$ slack.**

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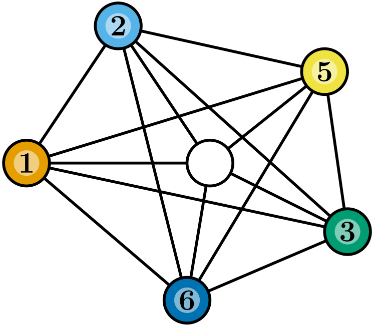

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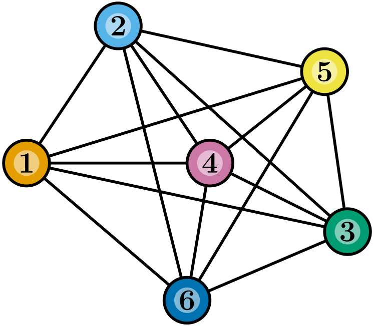

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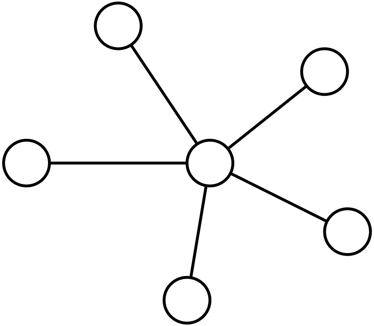

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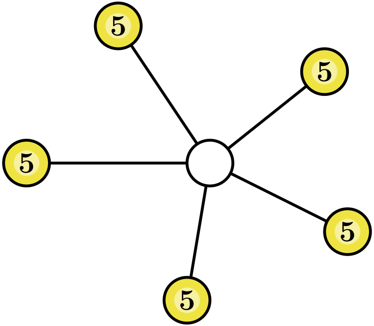

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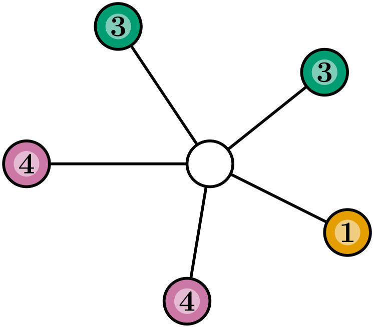

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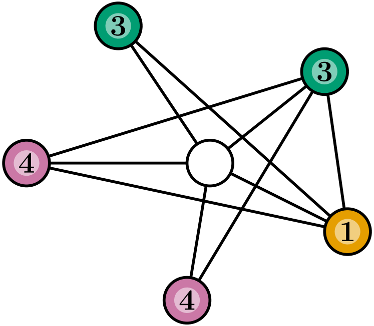
Nodes of the line-graph L_G ($e \in E_G$ iff $v_e \in V_{L_G}$; $v_e v_{e'} \in E_{L_G}$ iff $e = uv, e' = uv'$) have sparsity $\Omega(\Delta)$. **In an edge-coloring setting, [EPS15]'s result gives $\Omega(\Delta)$ slack.**

Creating slack: sparsity



Intuition: if your neighbors are mostly disconnected, it's likely some of them will pick the same color.

Formally: $\zeta_v = \frac{1}{\Delta}\left(\binom{\Delta}{2} - |E[N(v)]|\right)$.

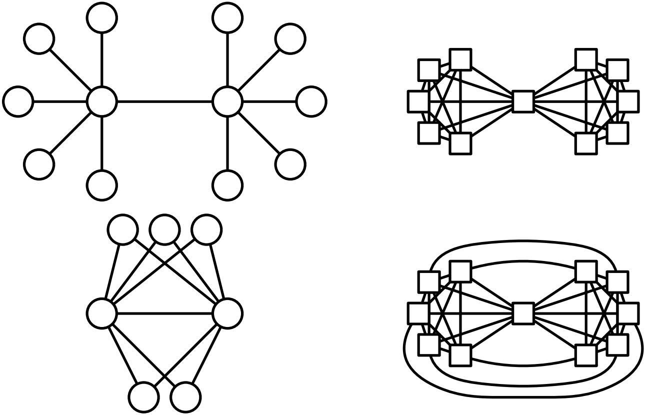
Theorem ([EPS15]): Let each node try a random color, then afterwards:

$$\Pr[s_v \le c \cdot \zeta_v] \le \exp(-c' \cdot \zeta_v).$$

for some universal constants c and c'.

Nodes of the line-graph L_G ($e \in E_G$ iff $v_e \in V_{L_G}$; $v_e v_{e'} \in E_{L_G}$ iff $e = uv, e' = uv'$) have sparsity $\Omega(\Delta)$. **In an edge-coloring setting, [EPS15]'s result gives $\Omega(\Delta)$ slack.**

Inherent sparsity of line-graphs



Pairwise-Independent Hash Functions

Definition ((ϵ-almost) Pairwise-Independent Hash Functions)

A set $\mathcal{H}$ of functions $h : [N] \to [M]$ s.t.:
$\forall x \neq y \in [N]^2, z, z' \in [M]^2$,
$|\Pr_{h \in \mathcal{H}}[h(x) = z \wedge h(y) = z'] - 1/M^2| \leq \epsilon/M^2$.

Let p be a prime such that $p \geq M^3 \cdot \log_p N/\epsilon$.

Decompose x into basis p, i.e., compute $x_1, \ldots, x_{\lceil \log_p N \rceil}$ such that $x = \sum_i x_i \cdot p^i$.

Consider the polynomial $P_x(X) = \sum_i x_i \cdot X^i$, of degree $\leq \log_p N$.

Pick a, b, z uniformly at random in $\{0, \ldots, p-1\}$.

Compute: $h_{a,b,z} = ((a \cdot P_x(z) + b) \mod p) \mod M$.

Finite fields of size p^k, p prime

Example with $p = 2$.

Pick a degree k polynomial irreducible over $\mathbb{F}_2$. For $p = 2$, $P(X) = X^k + X + 1$ always works: $P(0) = P(1) = 1 \mod 2$.

Consider the $\mathbb{F}_2[X]/P(X)$, i.e., polynomials over $\mathbb{F}_2$ mod P.

It is equivalent to defining that there's an element a s.t. $a^k = 1 + a$, and considering the 2^k possible words that can be written as combinations of $1, a, \ldots, a^{k-1}$.

Example of a multiplication over this field: let $k = 3$, $(1 + a^2) \times a = a + a^3 = a + 1 + a = 1$.